UNIQUENESS OF THE FINITE-DIFFERENCE SOLUTION OF THE INVERSE PROBLEM OF HEAT CONDUCTION

A. M. Denisov and A. G. Orlov

The article examines the problem of determination of the coefficients of heat conduction and heat capacity from a system of difference equations for the equation of heat conduction with additional information about the solution of the difference problem. Uniqueness of the solution of the inverse problem is proved.

In studies of various thermophysical processes, it is often important to be able to determine the thermophysical characteristics of the medium from the observation results. In mathematical terms, this constitutes the problem of determination of the coefficients of the equation of heat conduction given additional information about the solution of some mixed problem for this equation. Inverse problems of this kind have been discussed by many authors (see, e.g., [1-3]).

An alternative formulation can be proposed for the inverse problems of heat conduction, when the inverse problem is posed for a system of difference equations approximating some differential problem for the equation of heat conduction [4]. In this formulation, the problem involves determination of the coefficients of a difference operator. Some problems of this kind have been studied in [5].

Consider the inverse problem for the system of difference equations

\[
\begin{align*}
\frac{u_i^j}{k} &= \frac{\tau}{k^2} \left\{ \frac{k_{i+1/2}}{c_i} (u_{i+1}^{j+1} - u_i^{j+1}) - \frac{k_{i-1/2}}{c_i} (u_i^{j+1} - u_{i-1}^{j+1}) \right\} + u_i^{j-1}, \\
i &= 1, \ldots, N - 1; \quad j = 1, \ldots, M; \\
k_{i-1/2} > 0, \quad i = 1, \ldots, N; \quad c_i > 0, \quad i = 1, \ldots, N - 1; \\
u_i^0 &= 0, \quad i = 0, \ldots, N; \\
u_0^j &= 0, \quad j = 1, \ldots, M; \\
u_N^j &= \varphi^j, \quad j = 1, \ldots, M; \quad \varphi^0 = 0, \quad \varphi^j \neq 0; \\
u_N^j - u_{N-1}^j &= \frac{\mu^j}{k_{N-1/2}}, \quad j = 1, \ldots, M,
\end{align*}
\]

where \(\mu^j\) and \(\varphi^j\) for \(j = 1, \ldots, M\) are known and it is required to determine \(k_{1-1/2}, c_i, u_i^1\).

The objective of this study is to find the number of time layers \(M\) required for unique determination of \(k_{1-1/2}, c_i, u_i^1\).

**THEOREM 1.** If \(M = 2N - 1\), then system (1)-(5) has at most one solution for the unknowns \(k_{1-1/2}, c_i, u_i^1\), \(i = 1, \ldots, N, c_i, i = 1, \ldots, N - 1, u_i^1, i = 0, \ldots, N, j = 0, \ldots, M\), from (1)-(5).

**Proof.** Let \(k_{1-1/2}, c_i, i = 1, \ldots, N, k_{1-1/2} > 0, c_i, i = 1, \ldots, N - 1, c_i > 0, u_i^1, i = 0, \ldots, N, j = 0, \ldots, 2N - 1\) be a solution of system (1)-(5). We will show that \(k_{1-1/2}, c_i, u_i^1\) are determined uniquely.

From (1)-(5) we obviously have

\[
u_i^1 = 0, \quad i = 0, \ldots, N - 1, \quad u_N^j = \varphi^j \neq 0.
\]
From (4)-(6) we obtain $u_{N-1} = \varphi^1 - h\mu^1/k_{N-1/2}$. Hence $k_{N-1/2} = h\mu^1/\varphi^1$. We rewrite Eq. (1) in a different notation:

$$u_i^j = \frac{\tau}{h^2} \left\{ a_i (u_{i+1}^{j-1} - u_i^{j-1}) - b_i (u_i^{j-1} - u_{i-1}^{j-1}) \right\} + u_i^{j-1},$$  \hspace{1cm} (7)

where

$$a_i = \frac{k_{i+1/2}}{c_i}, \quad b_i = \frac{k_i-1/2}{c_i}, \quad i = 1, \ldots, N-1. \hspace{1cm} (8)$$

We will show that $a_i$, $b_i$, $i = 1, \ldots, N-1$, are determined uniquely from (7), (2)-(5).

Let $j = 2$. From (7), (3), (6) it follows that

$$u_i^2 = 0, \quad i = 0, \ldots, N-2,$$
$$u_{N-1}^2 = \frac{\tau}{h^2} a_{N-1} u_N^2 = \frac{\tau}{h^2} a_{N-1} \varphi^1 \neq 0. \hspace{1cm} (10)$$

From (4), (5) we have

$$u_{N-1}^2 = \varphi^2 - \frac{h\mu^2}{k_{N-1/2}}, \hspace{1cm} (11)$$

and from (10), (11)

$$a_{N-1} = \frac{\varphi^2 - \frac{h\mu^2}{k_{N-1/2}}}{h^2 \varphi^1}. \hspace{1cm} (12)$$

Thus, $a_{N-1}$ is determined uniquely. From (4), (11) it follows that $u_N^2$ and $u_{N-1}^2$ are determined uniquely and $u_{N-1}^2 \neq 0$. Let $j = 3$. From (7), (3), (9) we obtain $u_i^3 = 0$, $i = 0, \ldots, N-3$, and from (4), (5)

$$u_{N-1}^3 = \varphi^3 - \frac{h\mu^3}{k_{N-1/2}}, \hspace{1cm} (12)$$

Using (9), we write (7) for $i = N - 1$, $j = 3$:

$$u_{N-1}^3 = \frac{\tau}{h^2} \left\{ a_{N-1} (u_N^3 - u_{N-1}^3) - b_{N-1} u_{N-1}^3 \right\} + u_{N-1}^3. \hspace{1cm} (13)$$

From (12), (13) we obtain

$$b_{N-1} = \frac{a_{N-1} (u_N^3 - u_{N-1}^3) + \frac{h}{2} (u_N^3 - \varphi^3 + \frac{h\mu^3}{k_{N-1/2}})}{u_{N-1}^3}. \hspace{1cm} (14)$$

Thus, $b_{N-1}$ is determined uniquely. From (7) written for $i = N - 2$, $j = 3$ and (9) we obtain

$$u_{N-2}^3 = \frac{\tau}{h^2} a_{N-2} u_{N-1}^3 \neq 0. \hspace{1cm} (14)$$

Let us find the equation for $a_{N-2}$. Consider (7) for $i = N - 1$, $j = 4$. Using equality (14), we obtain

$$u_{N-1}^4 = \frac{\tau}{h^2} \left\{ a_{N-1} (u_N^4 - u_{N-1}^4) - b_{N-1} (u_{N-1}^4 - u_{N-2}^4) \right\} + u_{N-1}^4 =$$
$$= \frac{\tau}{h^2} \left\{ a_{N-1} (u_N^4 - u_{N-1}^4) - b_{N-1} (u_{N-1}^4 - \frac{\tau}{h^2} a_{N-2} u_{N-1}^3) \right\} + u_{N-1}^4 =$$
$$= Ra_{N-2} + S, \quad R = \frac{\tau^2}{h^3} b_{N-1} u_{N-1}^2 \neq 0,$$
$$S = \frac{\tau}{h^2} \left\{ a_{N-1} (u_N^3 - u_{N-1}^3) - b_{N-1} u_{N-1}^3 \right\} + u_{N-1}^3. \hspace{1cm} (15)$$