The classical Young theorem states that if \( 1/r + 1 = 1/p + 1/q \), then the convolution operator from \( L_p \times L_q \) into \( L_r \) is bounded. Hörmander [1] has shown that if \( p > r \), then there is no translation-invariant bounded operators from \( L_p \) into \( L_r \) different from zero. Krein and Semenov have shown (see [2]) that if a symmetric space \( E \) and a measurable function \( \varphi \) are such that \( \| \sigma \|_E = o(t^{
u'}) \) as \( t \to 0 \) and \( \| \varphi \|_E = o(t) \) as \( t \to \infty \), then the convolution operator from \( E \times M(\varphi) \) into \( F \) is bounded, where \( \| x \|_F = \| x^{**}(t) \|_q \). The following problem arises naturally: How should the spaces \( E_1 \) and \( E_2 \) be connected to bring about the existence of a nonzero bounded convolution operator from \( E_1 \) into \( E_2 \)? This problem is new also for the spaces \( L_p \). In the scale of the spaces \( L_p \) the solution of this problem follows from Young's theorem. Indeed, if \( p < r \), then \( 1 > 1/p - 1/r > 0 \). Denote by \( q > 1 \) such a number that \( 1/p - 1/r = 1 - 1/q \). Then from the Young's theorem it follows that the convolution operator from \( L_p \) into \( L_r \) is bounded with the kernel \( K \in L_q \).

This problem has been solved by the author [3] for the class of symmetric spaces. This problem extends in a natural way to the following one: How should three symmetric spaces \( E_1, E_2, \) and \( E_3 \) be connected so that the integral convolution operation from \( E_1 \times E_3 \) into \( E_2 \) is bounded?

We introduce the necessary definitions.

**Definition 1.** A Banach space \( E \) of measurable functions on \([0, a]\) is called symmetric if:
1) from \( |x(t)| \leq |y(t)| \) for almost all \( t \in [0, a] \) and \( y \in E \) it follows that \( \| x \|_E \leq \| y \|_E \);
2) from the equimeasurability of \( x(t) \) and \( y(t) \) it follows that \( \| x \|_E = \| y \|_E \).

The number \( a \) may be finite or infinite.

**Definition 2.** The Lorentz and Marcinkiewicz spaces are defined as follows:

\[
\Lambda(\varphi) = \left\{ x(t) \left| \| x \|_{\Lambda_1} = \int_0^a x^*(t) d\varphi(t) < \infty \right. \right\},
\]
\[
M(\varphi) = \left\{ x(t) \left| \sup_{0 \leq t \leq a} \frac{1}{\varphi(t)} \int_0^t x^*(s) \, ds < \infty \right. \right\},
\]

where \( \varphi(t) \) and \( \varphi(t) \) are increasing concave functions on \([0, a] \).

**Definition 3.** The dilatation operator (for a similar transformation of the argument see, for example, [2]) is defined as follows:

\[
\sigma_t x(t) = \begin{cases} 
\frac{x(t)}{t}, & \text{if } t > 1, 0 \leq t \leq a, \\
0, & \text{if } t < 1, t \cdot a \leq t \leq a.
\end{cases}
\]

**Definition 4.** The Boyd's indices (see, for example, [2, 3]) \( \alpha_E \) and \( \beta_E \) are defined by

\[
\alpha_E = \lim_{\tau \to 0} \frac{\ln \| \sigma_\tau \|_{E \rightarrow E}}{\ln \tau},
\]
\[
\beta_E = \lim_{\tau \to \infty} \frac{\ln \| \sigma_\tau \|_{E \rightarrow E}}{\ln \tau}.
\]

The operation ** is defined by the following equality: \( x^{**} = \int_0^t x^*(s) \, ds \), where \( x^*(t) \) is the rearrangement of \( |x(t)| \) on \([0, a]\) in the decreasing order.
The convolution operator with the kernel $K(t)$ is defined by

$$T_x(t) = \begin{cases} \int_0^a K(t-s)x(s)\,ds, & a \leq t, \\ \int_0^a K(t-s)x(s)\,ds, & 0 \leq t < a. \end{cases}$$

To solve the mentioned problem we need some additional results.

**Lemma 1.** Let the symmetric spaces $E_1$ and $E_2$ be such that $\|x\|_{E_1} \leq E_2$. Then the inclusion $E_1 \subset E_2$ holds, where $\|x\|_{E_2} \leq E_1$. In particular, $E_1 \subset E_2$.

**Proof.** We have $\|x\|_{E_2} \leq \|x\|_{E_1}$ for $x \in E_2$. The lemma has been proved.

**Remark 1.** The statement of Lemma 1 is stronger than Theorem 5.6 from [1] since, instead of the condition $\|x\|_{E_1} \leq E_2$, we obtain the stronger one $\|x\|_{E_2} \leq E_1$ (which follows from the inclusion $E_1 \subset E_2$).

**Lemma 2.** Let $E_1$, $E_2$ be two symmetric spaces on $[0, 1]$ such that $\beta_{E_1} < \alpha_{E_2}$. Then we have the inclusion $E_1 \subset E_{2,1}$.

**Proof.** Considering the inclusion $E_1 \subset E_{2,1}$, we have

$$\|x\|_{E_1} \leq \int_0^1 \|x\|_{E_2} \,d\varphi \leq \alpha_{E_2} \|x\|_{E_1}.$$

The last integral is finite by virtue of the condition $\beta_{E_1} < \alpha_{E_2}$ and the properties of semi-multiplicative functions [1]. Now it is sufficient to apply Lemma 1. Lemma 2 has been proved.

**Theorem 1.** Let $E_1$, $E_2$, $E_3$ be symmetric spaces on $[0, 1]$ such that $1) \beta_{E_1} + \beta_{E_2} \leq 1 + \alpha_{E_2}; 2) \alpha_{E_1} + \beta_{E_2} > 1$. Then the integral convolution operator from $E_1 \times E_2$ into $E_3$ is bounded.

**Proof.** From condition 1) of the theorem it follows that there exists a number $\varepsilon > 0$ such that

$$\varepsilon + \beta_{E_1} + \beta_{E_2} < 1 + \alpha_{E_2}; \quad \alpha_{E_1} + \beta_{E_2} + \varepsilon > 1.$$  \hfill (1)

Further, let us take a function $\varphi$ such that $\beta_{E_1} = \alpha_{E_1} = \varepsilon + \beta_{E_2}$. Obviously the inequalities $\beta_{E_1} < \alpha_{E_1}$ and $\alpha_{E_1} + \varepsilon > 1$ are fulfilled. From condition 2) of Theorem 1, and from the theorem of Krein and Semenov on convolution operators (see [2], Theorem 6.16, p. 201), it follows that the convolution operator $T$ acting from $E_1 \times M(\varphi)$ into $E_{1,\varphi}(t)/\varphi$ is bounded.

It is easy to verify the inequality $\|x\|_{E_1,\varphi} \leq \chi(t) \|x\|_{E_2}$. From this, by the definition of the upper index of dilatation for the function $\|x\|_{E_2}$ and by condition 1) of Theorem 1, the inequality

$$\beta_{E_{1,\varphi}(t)/\varphi} \leq \beta_{\varphi} \leq 1 + \beta_{E_1} < \alpha_{E_2}$$

follows.

From Lemma 1 it follows that $E_{1,\varphi}(t)/\varphi \subset E_2$. Hence the operator $T$ acting from $E_1 \times M(\varphi)$ into $E_2$ is bounded. Considering the inequality $\beta_{E_2} < \alpha_{E_2}$ and Lemma 2, we obtain $E_3 \subset M(\varphi)$. Eventually we get $T: E_1 \times E_2 \rightarrow E_3$. The theorem has been proved.

**Theorem 2.** Let the symmetric spaces $E_1$, $E_2$, $E_3$ of measurable functions defined on $[0, 1]$ be such that $1) \alpha_{E_1} + \alpha_{E_2} = 1 + \alpha_{E_3}; 2) \alpha_{E_1} + \alpha_{E_2} > 1$. Then there exists a function $\varphi$ such that $M(\varphi) \subset E_3$ and the convolution operator from $E_1 \times M(\varphi)$ into $E_2$ is bounded.

**Proof.** Take a number $\varepsilon > 0$ such that $\alpha_{E_1} + \alpha_{E_2} - \varepsilon > 1$. Take a function $\varphi(t)$ (for example, a step function) such that $\beta_{\varphi} = \alpha_{\varphi} = \alpha_{E_1} - \varepsilon$. Obviously

$$\beta_{\varphi} < \alpha_{E_1}; \quad \beta_{\varphi} + \beta_{E_1} < 1 + \alpha_{E_2}; \quad \alpha_{E_1} + \alpha_{\varphi} > 1.$$