A solution method is proposed for the problem of allocating a given quantity of resource \( A \) on network graphs with two uncertain factors. One of the uncertain factors is the type of the network graph and the other is the capacity of the arcs in the graph.

Let a number of network graphs be given with events \( Z_1(k_1), \ldots, Z_n(k_i) \) and jobs \( I_{j_k}(k_i), k_1 = 1, \ldots, r_1 \). To each job is associated a completion time as a function of the allocated resource \( \phi_{j_{k_1}}^i(z_{j_{k_1}}^{(k_1)}, k_2), k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \). The given completion deadline for all jobs is \( T > 0 \). Events occur in the following sequence: first the operator selects the quantity of the resource \( A \). Given the resource quantity, the opponent or nature chooses the network graph, i.e., the number \( k_1, k_1 = 1, \ldots, r_1 \). Then the operator chooses the resource allocation \( z^{(k_1)} = (z_{j_{k_1}}^{(k_1)}, \ldots, z_{m_{k_1}}^{(k_1)}) \), \( z_{j_{k_1}}^{(k_1)} \geq 0, \sum_{j_{k_1}=1}^{m_{k_1}} z_{j_{k_1}}^{(k_1)} \leq A \). After that the opponent or nature chooses the values of the uncertain factors \( k_2, k_2 = 1, \ldots, r_2 \).

The choice of all these factors (vectors and numbers) determines the type of the network graph and the arrival times of all jobs. The occurrence times of the events are thus determined by the schedule \( t(k_1, k_2) = (t_{j_{k_1}}^{(k_1)}, k_2), k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \).

The problem is formulated as follows: find the minimum quantity of resource \( A^* \) which ensures completion times not exceeding \( T \) for all jobs for any values of \( k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \). It is assumed that the uncertain factors take the worst possible values for the choice of the resource quantity. Following [1], we represent the problem in the form

\[
\min_{A} \max_{k_1} \max_{k_2} \min_{A} \max_{k_1} \min_{k_2} \min_{k_1} \min_{k_2} A,
\]

\[
t^{(k_1, k_2)} - t^{(k_1, k_2)} \leq T,
\]

\[
t^{(k_1, k_2)} - t^{(k_1, k_2)} \geq \phi_{j_{k_1}}^i(z_{j_{k_1}}^{(k_1)}, k_2),
\]

\[
x_{j_{k_1}}^{(k_1)} \geq 0, \sum_{j_{k_1}=1}^{m_{k_1}} x_{j_{k_1}}^{(k_1)} \leq A,
\]

\[
s_{k_1} = 1, \ldots, m_{k_1}; k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2.
\]

This would be a particular case of the problem considered by Davydov and Khersonskaya if the constraint set were compact.

We will show that the constraint set can be compactified while preserving the optimal solution. The constraint set in this problem may be empty, and we can thus choose functions \( \phi_{j_{k_1}}^i(x_{j_{k_1}}^{(k_1)}, k_2) \), such that for any \( A \) it is impossible to satisfy the first \( r_1 \times r_2 \) constraints \( t^{(k_1, k_2)} - t^{(k_1, k_2)} \leq T \), e.g., for small \( T \) and large \( \phi_{j_{k_1}}^i(x_{j_{k_1}}^{(k_1)}, k_2) \) for any \( x_{j_{k_1}}^{(k_1)} \). In what follows we assume that all the functions \( \phi_{j_{k_1}}^i(x_{j_{k_1}}^{(k_1)}, k_2) \) are continuous in all \( x_{j_{k_1}}^{(k_1)} \) for \( j_{k_1} = 1, \ldots, m_{k_1}; k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \).

Now assume that there exists \( \tilde{A} > 0 \) such that the constraint system (1) is nonempty for \( \tilde{A} \). This \( \tilde{A} \) may be nonoptimal. Then the optimal \( A^* \leq \tilde{A} \) is upper bounded by \( \tilde{A} \). Take some \( A \geq 0 \). Suppose that a feasible schedule for \( A \) is \( x^{(k_1)}, \{t^{(k_1, k_2)}\} \). If we fix \( k_1 = 1 \) and \( k_2 = 2 \), then

\[
\min_{j_{k_1}} (t^{(k_1, k_2)} - t^{(k_1, k_2)}) \geq \min_{j_{k_1}} (t^{(k_1, k_2)} - t^{(k_1, k_2)}),
\]

where minimization is over the constraint system.
of problem (1) with \( x^{k(i)} \) replaced by \( x^{k(i)} \). By the main theorem of network scheduling, the last minimum is achieved on the schedule defined by this theorem. Denote this schedule by \( \bar{t}_{i}^{k_1,k_2} \), where \( \bar{t}_{i}^{k_1,k_2} = 0 \). For such schedules we have the bound

\[
0 \leq \bar{t}_{i}^{k_1,k_2} \leq \sum_{j_{k_1}=1}^{m_{k_1}} \phi^{(k_1)}(x^{k_1}_{j_{k_1}}, k_2) \leq \max_{1 \leq k_2 \leq r_2} \sum_{j_{k_1}=1}^{m_{k_1}} \sum_{k_1=1}^{r_1} \phi^{(k_1)}(x^{k_1}_{j_{k_1}}, k_2) = K,
\]

\[
\sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}} = A, \quad x^{k_1}_{j_{k_1}} \geq 0, \quad k_1 = 1, \ldots, r_1.
\]

Now take the combination \( \bar{A}, \bar{x}^{(k_1)} \), \( \{\bar{t}^{k_1,k_2}\} \) for \( k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \). This combination is obviously a feasible combination subject to the constraints (1) and moreover \( 0 \leq \bar{t}^{k_1,k_2} \leq K \). The objective function value is preserved, and therefore for each feasible \( A \) the constraint set of problem (1) is \( 0 \leq \bar{t}^{k_1,k_2} \leq K \). The constraint set of feasible combinations of problem (1) thus becomes a nonempty compactum, and the problem reduces to a particular case of the multiple minimax problem with coupled constraints, previously considered by Davydov and Khersonskaya.

Let us describe the solution procedure. In our notation, \( v \) is \( A \), \( y^{(k_1)} \) is \( x^{k(i)} \), \( y^{(k_1,k_2)} \) is \( t^{k_1,k_2} \). We consider five-fold optimization. In accordance with the previous argument, we first solve the problem

\[
\min_{A \in \{x^{k_1}, y^{(k_1)}, y^{(k_1,k_2)}\}} A,
\]

\[
t^{k_1,k_2} - \bar{t}^{k_1,k_2} \leq T,
\]

\[
x^{k_1}_{j_{k_1}} \geq 0, \quad \sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}} \leq A^*,
\]

\( s_{k_1} = 1, \ldots, m_{k_1}; k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \).

This problem has the solution \( A^*, \{\bar{x}^{(k_1)}, \bar{t}^{k_1,k_2}\} \). In the next stage, we solve \( r_1 \) nonlinear programming problems

\[
\min_{x^{(k_1)}, \{y^{(k_1)}, y^{(k_1,k_2)}\}} A^*,
\]

\[
t^{k_1,k_2} - \bar{t}^{k_1,k_2} \leq T,
\]

\[
x^{k_1}_{j_{k_1}} \geq 0, \quad \sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}} \leq A^*,
\]

\( s_{k_1} = 1, \ldots, m_{k_1}; k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \).

Its solution is the combination \( \nu^*_1, \{\bar{x}^{(k_1)}, \bar{t}^{k_1,k_2}\} \). The next stage is to solve the \( r_1 \) nonlinear programming problems

\[
\min_{x^{(k_1)}} \sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}},
\]

\[
\sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}} \leq A^*,
\]

\[
t^{k_1,k_2} - \bar{t}^{k_1,k_2} \leq T,
\]

\[
x^{k_1}_{j_{k_1}} \geq 0, \quad \sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}} \leq A^*,
\]

\( s_{k_1} = 1, \ldots, m_{k_1}; k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \).

We obtain the optimal resource allocation \( x^{\nu_1^*(k_1)} \). Then we solve \( r_1 \times r_2 \) linear programming problems

\[
\min_{x^{\nu_1^*(k_1)}, A^*} \sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}},
\]

\[
t^{k_1,k_2} - \bar{t}^{k_1,k_2} \leq T,
\]

\[
x^{k_1}_{j_{k_1}} \geq 0, \quad \sum_{j_{k_1}=1}^{m_{k_1}} x^{k_1}_{j_{k_1}} \leq A^*,
\]

\( s_{k_1} = 1, \ldots, m_{k_1}; k_1 = 1, \ldots, r_1; k_2 = 1, \ldots, r_2 \).