HYDRAULIC NETWORK DESIGN FOR STATIONARY OPERATION

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A hydraulic network design method is proposed, which uses design parameters ensuring constant discharges through the sections of the network. For tree type networks, the weight minimization problem is reduced to a mathematical programming problem.

1. Consider a hydraulic network with $l + 1$ leaves $A_0, A_1, ..., A_l$. From capacities provided at nodes $A_1, ..., A_l$ (tanks, reservoirs) the liquid is pumped to the capacity at node $A_0$. From node $A_0$ the liquid is supplied at a constant discharge rate $Q_0$ to the user.

Let $J = \{0, 1, ..., l, ..., m\}$ be the index set of the network sections. Sections $0, 1, ..., l$ are adjacent to the corresponding nodes $A_0, A_1, ..., A_l$. The lengths and the locations of the network sections, and also the type of pumps and the shape and volume of the capacities at the points $A_0 = A_1, ..., A_l$ are given and fixed. We can only alter the diameters of the network sections $d_j, j \in J$. It is required to design a minimum-weight network. This is usually done under additional constraints. For instance, we may stipulate that the capacities at nodes $A_1, ..., A_l$ all empty simultaneously.

Identify the spanning tree $\mathcal{D}$ that contains all the network nodes. Let $J_1 = \{0, 1, ..., n\}, n \geq l$ be the index set of the sections included in the tree $\mathcal{D}$. We assume that a certain direction has been chosen for all network sections. For the tree $\mathcal{D}$ positive direction corresponds to motion from the nodes $A_i, i = 1, ..., l$, to the node $A_0$. The discharge $Q_j$ of the liquid through section $j$ is assumed positive if the liquid moves in the positive direction; otherwise the discharge $Q_j$ is negative.

The discharge rates $Q_j(t)$ satisfy a system of ordinary differential equations (see below), which has a unique stationary point.

Section 2 proposes a network design principle "for stationary operation." In Sec. 3 this principle is applied to tree type networks to reduce the weight minimization problem to a mathematical programming problem.

2. Assume that the energy loss by the liquid in section $j$ is $s_j Q_j |Q_j|$, where $s_j = k_j/d_j^4$, and $d_j$ is the flow diameter [1]. Let $g_i$ be the current weight of the liquid stored at node $A_i, i = 0, 1, ..., l$, and assume that the liquid at this node is a cylinder with base area $F_i$ and current height $h_i = g_i/\gamma F_i + h_i$, where $h_i$ is the height of the lower base of the storage capacity relative to some fixed level and $\gamma$ is the specific weight of the liquid.

Assume that the pressure characteristic of the pump at node $A_i$ is given by the function $a_i - b_i Q_i |Q_i|$, where $a_i, b_i > 0, i = 1, ..., l$. Let us write the Kirchhoff equation which determines the discharges for the network sections at any moment in time.

We start with the necessary notation. Let $J_2 = J \cup J_1$ be the index set of network sections not included in the tree $\mathcal{D}$. Augment each section $j \in J_2$ by sections from the tree $\mathcal{D}$ so as to form a cycle $C_j$. For each $j \in J_1$ denote by $M_j$ the set of indices $p \in J_2$ of network sections whose end nodes (in accordance with the specified direction) are on the paths leading from the nodes $A_i, 1 \leq i \leq l$, to section $j$. Similarly for each $j \in J_1$ denote by $N_j$ the set of indices $p \in J_2$ of network sections whose initial nodes are on the paths leading from the nodes $A_i, 1 \leq i \leq l$, to section $j$. For each node $A_i, 1 \leq i \leq l$, denote by $K_i$ the set of indices $j \in J_1$ of the sections of the tree $\mathcal{D}$ that are included in the path from node $A_i$ to node $A_0$. Finally, for each $j \in J_1$, let $E_j = \{i | 1 \leq i \leq l, j \in K_i\}.$

The system of Kirchhoff equations can be written in the form (see, e.g., [1])

$$\frac{g_i}{\gamma F_i} + h_i + a_i - b_i Q_i |Q_i| = \sum_{j \in K_i} s_j Q_j |Q_j| + \frac{g_0}{\gamma F_0} + h_0, \quad i = 1, ..., l.$$
In what follows, we consider $Q_j$, $j \in J$, and $g_i$, $i = 0, 1, \ldots, l$, as functions of time. From their definition we have the equations

$$\frac{d}{dt} Q_i = -Q_i, \quad i = 1, \ldots, l,$$

$$\frac{d}{dt} g_0 = \sum_{i=1}^l Q_i - Q_0.$$  

Differentiating Eqs. (1)-(3) with respect to time and using (4), we obtain a system of differential equations for $Q_j(t)$, $j \in J$:

$$b_i |Q_i| Q_i + \sum_{j \in J} s_j |Q_j| Q_j = \frac{Q_0 - \sum_{i=1}^l Q_i}{2F_0} - \frac{Q_i}{2F_i}, \quad i = 1, \ldots, l,$$

$$\sum_{j \in J} s_j |Q_j| Q_j = 0, \quad p \in J_2,$$

$$\dot{Q}_j = \sum_{i \in E_j} \dot{Q}_i - \sum_{p \in N_j} \dot{Q}_p + \sum_{p \in M_j} \dot{Q}_p, \quad j \in J_1.$$  

Let $J' = J_2 \cup \{1, \ldots, l\}$,

$$r = |J'| = |J_2| + l, \quad Q = (Q_j, \quad j \in J'),$$

$$\dot{Q} = (\dot{Q}_j, \quad j \in J'), \quad \Pi = (\Pi_j, \quad j \in J'),$$

$$\Pi_j = -\frac{\sum_{i=1}^l Q_i - Q_0}{2F_0} - \frac{Q_j}{2F_j}, \quad 1 \leq j \leq l,$$

$$\Pi_j \equiv 0, \quad j \in J_2.$$  

For $1 \leq i, p \leq l$, define the sets

$$K_{ip} = K_i \cap K_p, \quad K_{ii} = K_i.$$

Using (7), eliminate from system (5), (6) $\dot{Q}_j$, $j \in J'$. As a result we obtain the system of equations

$$\Psi \dot{Q} = -\Pi,$$

where $\Psi$ is the $r \times r$ matrix with the following elements:

$$\psi_{ii} = b_i |Q_i| + \sum_{j \in J} s_j |Q_j|, \quad 1 \leq i \leq l,$$

$$\psi_{ip} = \psi_{pi} = \sum_{j \in K_{ip}} s_j |Q_j|, \quad 1 \leq i \neq p \leq l,$$

$$\psi_{ip} = \psi_{pi} = \sum_{j \in J_p} s_j |Q_j| - \sum_{j \in J_p} s_j |Q_j|, \quad 1 \leq i \leq l, \quad p \in J_2,$$

$$\psi_{pp} = s_p |Q_p| + \sum_{j \in J_p \cup N_j} s_j |Q_j|, \quad p \in J_2.$$  

Note that for $p \in J_2$, we either have $M_j = 0$ for all $j \in C_p$ or $N_j = 0$ for all $j \in C_p$. Moreover, if we use (3) to eliminate from system (8) $Q_j$, $j \in J'$, we obtain a system of ordinary differential equations for $Q$. This system can be represented in a form solved for the derivatives on the basis of the following proposition.

**LEMMA.** The matrix $\Psi$ of system (8) is symmetric and positive definite.