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LITERATURE CITED


QUADRATIC SINGULARITIES OF THE PFaffIAN THETA DIVISOR OF A PRYM VARIETY

V. I. Kanev

We cite certain facts from the theory of Prym varieties, which we will use. All of them are contained in [1, 2]. Let $\pi: C \to \bar{C}$ be an unramified double covering of nonsingular curves and $i$ be the involution of $C$ corresponding to $\pi$.

The Prym variety $P$ is defined as follows:

$$P = \{x - i(x) \mid x \in J(\bar{C})\},$$

where $J(\bar{C})$ is the Jacobian of $\bar{C}$. A canonical principal polarization is defined on $P$ by the divisor $E$. If $D$ is an arbitrary divisor of degree $2g - 2$ on $C$ such that $NmD \subseteq |K_C|$, then

$$E = \{\xi \in J(\bar{C}) \mid L_\xi \simeq L(D), \quad \deg D = 2g - 2, \quad NmD \subseteq |K_C|, \quad \dim \Gamma(O(D)) = 0 \pmod{2} \& > 0\}.$$

Let us set $P_{2g-2} = P(T_{\bar{C}},)$, $P_{2g-2} = P(T, p)$, and $P_{2g-2} \supseteq P_{2g-2}$. Let $\sigma$ be a divisor on $C$, $2\sigma \sim 0$, corresponding to the covering $\pi$.

Then we have the natural identifications: $P_{2g-2} = |K_\bar{C}|^*$ and $P_{2g-2} = |K_C(\sigma)|^*$. Let $\xi \equiv \text{sign} E$. After displacement and projection, we can consider the tangent cones $Q$ and $q$ at $\xi$ respectively to $\Theta$ and $E$ as subvarieties in $P_{2g-2}$ (respectively, in $P_{2g-2}$). The equations of $Q$ and $q$ are obtained in the following manner: The definition of $P$ gives an isomorphism $\Theta: L_\xi \otimes iL_\xi \to K_\bar{C}$. We define a pairing $\langle \cdot, \cdot \rangle$ and its extension $\mu$ to the tensor product:

$$\langle \cdot, \cdot \rangle: \Gamma(L_\xi) \times \Gamma(L_\xi) \to \Gamma\left(\Omega_\bar{C}^g\right), \quad \langle s, t \rangle = \Phi(s \otimes it);$$

$$\mu: \Gamma(L_\xi) \otimes \Gamma(L_\xi) \to \Gamma\left(\Omega_\bar{C}^g\right), \quad \mu(s \otimes t) = \langle s, t \rangle.$$

Since $i^*\langle s, t \rangle = \langle t, s \rangle$, it follows that the restriction of $\mu$ to $\Lambda^g\Gamma(L_\xi)$ maps $\Lambda^g\Gamma(L_\xi)$ into $\Gamma\left(\Omega_\bar{C}^g\right)$. If $s_1, s_2, \ldots, s_n$ is a basis of $\Gamma(L_\xi)$, the $Q_\xi$ is given in $P_{2g-2}$ by the equation

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and \( q_\xi \) is given by the equation \( Pf(\mu(s_i \setminus s_j))_{i \leq j \leq n} \) (Pf is the Pfaffian of the matrix).

**Definition.** If \( Pf(\mu(s_i \setminus s_j)) \neq 0 \), then \( \xi \) is called a Pfaffian singularity.

We consider the case \( n = 4 \), i.e., where \( q_\xi \) is a quadric.

**Lemma 1.** Let \( \xi \) be a quadratic Pfaffian singularity of \( \mathbb{E} \). Suppose that \( q \) is the tangent cone to \( \xi \). Then the following statements are valid:

(i) \( q = \bigcup_{s \in \mathbb{R}^2} \langle \text{div}(s) \rangle \), where \( \tau_s \) is the linear subspace of \( P^{s-2} \), defined by the equations \( \mu(s \setminus t) = 0 \), where \( t \in \Gamma(L_0) \).

(ii) The center of \( q \) is determined by the equations

\[
\mu(s \setminus t) = 0, \quad s, t \in \Gamma(L_0).
\]

**Proof.** (i) \( q = Q \cap P^{s-2} \). We know that (see [3]) \( Q = \bigcup_{s \in \mathbb{R}^2} \langle \text{div}(s) \rangle \), \( \langle \cdot \rangle \) means the linear hull \( \langle \text{div}(s) \rangle \) is defined by the equation \( \langle s, t \rangle = 0 \), \( t \in \Gamma(L_0) \). Consequently, \( \tau_s = \langle s \setminus t \rangle \cap P^{s-2} \) is defined by the equations \( \mu(s \setminus t) = \langle s, t \rangle = \langle t, s \rangle \).

(ii) Let \( \omega_{ij} = \mu(s_i \setminus s_j) q \) be given by the equation

\[
Pf(\omega_{ij}) = \omega_{12} \omega_{24} - \omega_{15} \omega_{34} + \omega_{14} \omega_{23} = 0.
\]

\( \omega_{ij} \) are the equations of the center of \( q \).

By virtue of Lemma 1, it is more natural to seek a geometrical description of the dual quadric \( \bar{q} \subset \{K_0(a)\} \). Let us consider \( E = \mathbb{R}^2 \cap L_0 \). This is a bundle of rank 2 on \( \mathbb{C} \), and, as proved in [2], \( \det E = K_0(a) \). By definition, \( \Gamma(L_0) = \Gamma(E) \), so that \( \mu \) is decomposed into the composition

\[
\mu : P(\Lambda^2 \Gamma(E)) \to P(\Gamma(\Lambda^2 E)) \cong K_0(a).
\]

Hence \( \mu \) is linear. If \( \xi \) is a quadratic singularity, then \( \dim \Gamma(L_0) = 4 \), and \( \dim \Lambda^2 \Gamma(L_0) = 6 \). Let us set \( P(\Lambda^2 \Gamma(L_0)) = P^6 \), and suppose that \( G = \text{Gr}(1, 3) \subset P^6 \) is the Grassmannian variety of straight lines in \( P^3 = P(\Gamma(L_0)) \). Let \( W = \text{Sing} q \subset P^{s-2} = |K_0(a)| \). Then it follows from Lemma 1 that \( \mu \) maps \( P^3 \) onto \( W \). The Grassmannian variety \( G \) contains two systems of planes: \( \alpha \)-planes consisting of the straight lines passing through a given point, and \( \beta \)-planes consisting of the straight lines contained in a given plane. It follows from Lemma 1 that \( \tau_s = \mu(s) \), where \( \tau_s \) is the \( \alpha \)-plane corresponding to the section \( s \in \Gamma(L_0) \).

The following lemma is needed to compute the tangent cone dual to a quadric.

**Lemma 2.** Let \( q \) be a quadric and \( q \subset P^n \). Suppose that each maximal subspace \( E \subset q \) has a dual \( E^\perp \subset \mathbb{P}^n \).

Then the following statements are valid:

(i) If \( r(kq) \) is odd, then \( \bar{q} = \bigcup E^\perp, E \subset q \), and \( E \) is maximal.

(ii) If \( r(kq) \) is even, then either two irreducible families of the subspaces \( E^\perp \), or only one, pass through each point \( x \in \mathbb{P}^n \).

**Proof.** We can obviously assume that the quadric is nonsingular. Let \( r \) be the mapping

\[
r : q \to q, \quad r(x) = T_x q.
\]

It is clear that \( r(E) \subset E^\perp \) for each subspace \( E \subset q \). If \( r(kq) = 2k + 1 \), and \( E \) is maximal, then \( \dim E = k \) and \( \dim E^\perp = k \); whence \( r(E) = E^\perp \). The statement (i) is proved.

If \( r(kq) = 2k \) and \( E \) is a maximal subspace of \( q \), then \( \dim E = k - 1 \), and \( \dim E^\perp = k \), so that \( r(E) \subset E^\perp \). We take a point \( h \subset \mathbb{P}^n \), \( h \) is a hyperplane in \( P^n \) and \( h \) cuts from \( q \) either a nonsingular quadric of rank \( 2k - 1 \) or a singular quadric of rank \( 2k - 2 \). In both the cases \( q \cap h = \bigcup_{E \subset q} h \) and \( E \) is maximal in \( q \).

In the first case, \( h \) does not touch \( q \) and \( q \cap h \) has two systems of generators. Consequently, there are two irreducible systems of the subspaces \( E^\perp \). In the second case, \( h \) touches \( q \) and \( q \cap h \) has only one system of generators.