If \( h_{n-1} < h < h_n \), then, taking into account the monotonicity of \( \sigma(h) \) and \( \alpha(x,h) \) with respect to \( h \), we obtain that
\[
\frac{\alpha(X(t), h) \sigma(\Delta h/4 \log \log h)}{h} \leq \frac{\alpha(X(t), h_n) \sigma(\Delta h_n/4 \log \log h_n)}{h_n},
\]
from where there follows the assertion of the theorem.

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LITERATURE CITED


ASYMPTOTIC EXPANSIONS FOR THE PROBABILITIES OF LARGE RUNS OF NONSTATIONARY GAUSSIAN PROCESSES

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1. The problems of the estimation and the determination of the asymptotics of the distribution of the maximum and the minimum moduli of a Gaussian process has been repeatedly investigated by several authors (see [1]). In [2] one has found the asymptotic expansions for the distributions of the maximum of a stationary process with sufficiently smooth trajectories. In the present note we give an other approach for the solution of this problem. We consider the case of a nonstationary process with smooth trajectories in the neighborhood of the point of absolute maximum of the variance.

2. Let \( X(t), t \in (-\infty, +\infty) \), be a real Gaussian random process. We denote \( \mathcal{M} X(t) = m(t), \mathcal{M} (X(t) - m(t)) (X(s) - m(s)) = r(t,s), D(X(t) - X(s)) = d^2(t,s), r(t,t) = \sigma^2(t), r_{ij}(t,s) = \frac{\partial^{|i+j|}}{\partial t^i \partial s^j} r(t,s), i \geq 0, j \geq 0, p_{x,y} \ldots (x,y, \ldots) \) is the density of the vector \((X, Y, \ldots)\) at the point \((x, y, \ldots)\). We shall assume that the following conditions hold:

I. The least upper bound of the function \( \sigma(t) \) is attained at the unique point \( t_0, 0 \geq \sigma^{(a)}(t_0) > -\infty \) for some integer \( l > 0 \), \( \sigma^{(a)}(t_0) = 0 \) for \( r < 2l \) and \( \limsup_{t \to \infty} \sigma(t) < \sigma(t_0) \).

II. There exist an integer \( n \geq 2 \) and real \( a \geq 0, C < \infty \) and \( \delta > 0 \) such that on the segment \([t_0 - \delta, t_0 + \delta]\) there exists the \( n \)-th derivative \( X^{(n)}(t) \) in the mean square (m.s.) process \( X(t) \) and \( \mathcal{M} (X^{(n)}(t) - X^{(n)}(s))^2 \leq C |t - s|^a \), while if \( a = 0 \), then \( X^{(n)}(t) \) is continuous on \([t_0 - \delta, t_0 + \delta]\) in m.s.

III. \( DX'(t_0) \geq 0 \).

IV. The Dudley integral for the process \( \dot{X}(t) = X(t) - m(t) \) converges and \( m(t) \) is a function bounded from above.

We recall that by the Dudley integral we mean the integral
\[ \Psi(\delta) = \int_0^\delta \ln N(\varepsilon)^{1/2} \, d\varepsilon, \]
where \( N(\varepsilon) \) is the power of the smallest \( \varepsilon \)-net in the pseudometric \( d(t, s) \) on the line \( (-\infty, +\infty) \).

**THEOREM.** If the Gaussian process \( X(t) \) satisfies the condition I-IV, then
\[ u^{1/2} \exp \left( \frac{u^2}{2\sigma^2(t_0)} \right) P \left( \sup_{t \in (-\infty, +\infty)} X(t) > u \right) = \sum_{k=0}^{n-2l} c_k u^{-k} + O(u^{-n+2l-\gamma}), \quad u \to \infty, \]
where the coefficients \( c_k, k \leq n - 2l \) depend only on the moments of the variables \( X^{(j)}(t_0), j \leq k + 2l \), they are defined below (Sec. 6), and if \( \alpha = 0 \), then \( O(\cdot) \) has to be replaced by \( o(\cdot) \).

In the case when \( m^{(k)}(t_0) = 0 \) for \( 0 < k < n \), the form of the coefficients \( c_k \) is given by the relation (8).

In the case when \( m(t_0) = 0, l = 1 \) then
\[
c_0 = \frac{\gamma(t_0) \sigma^2(t_0)}{\sqrt{2\pi} \sigma_1(t_0)} \exp \left( -\frac{m'(t_0)^2}{2\sigma^2(t_0)} + \frac{m^2(t_0)}{2\sigma_1^2(t_0)} \right) \frac{\sigma(t_0) \gamma(t_0) \gamma'(t_0)}{2 \gamma(0)^{3/2}} \exp \left( -\frac{m'(t_0) \sigma^2(t_0)}{2 \gamma(0)} \right). \]

Here and in the sequel,
\[
\gamma^2(t) = r_{11}(t, t), \quad \mu(t) = (\gamma(t) \sigma(t))^{-1} r_{01}(t, t), \\
a_1(t) = (2\sigma^2(t)(1 - \mu^2(t)))^{-1}, \\
a_1(t_0) = -\frac{(\gamma^2(t_0))_{t=t_0}}{\sigma^2(t_0)} + \frac{(\mu'(t_0))^2}{\sigma^2(t_0)}. \]

3. We shall assume that \( t_0 = 0, \sigma(t_0) = 1 \). Then, the general case is obtained easily by a change of scale. The method of the derivation of the above given asymptotic expansion is based on the following inequalities:

\[
0 \leq P \left( \sup_{|t| \leq \Delta} X(t) \geq u \right) - P \left( \sup_{|t| \leq \Delta} X(t) \geq u \right) \leq \frac{1}{2} \left( \alpha_2(u) + \beta_2(u) \right) + P \left( \sup_{|t| \leq \Delta} X(t) \geq u, X(\Delta) \geq u \right), \quad \Delta > 0; \tag{1}
\]

\[
0 \leq \mathcal{N}_u \left[ -\Delta, \Delta \right] + P \left( X(-\Delta) \geq u \right) - P \left( \max_{|t| \leq \Delta} X(t) \geq u \right) \leq \frac{1}{2} \left( \alpha_2(u) + \beta_2(u) \right) + P \left( X(-\Delta) \geq u, X(\Delta) \geq u \right), \tag{2}
\]

where \( \mathcal{N}_u = N_u \left[ -\Delta, \Delta \right] \) is the number of the exits of the process \( X(t) \) beyond the level \( u \) on the segment \( [-\Delta, \Delta] \), \( \alpha_2(u) = \mathcal{M}_u(N_u - 1), \beta_2(u) = \mathcal{M}_u(L_u - 1) \), \( L_u \) is the number of entries of the process under the level \( u \) on the segment \( [-\Delta, \Delta] \) (see [3]).

From conditions I and III there follows that for sufficiently small \( \Delta \), the quantities \( \mathcal{M}_u, \alpha_2(u) \) and \( \beta_2(u) \) are finite. Indeed, by virtue of I,
\[
0 = (\sigma^2(t))' \big|_{t=0} = r_{i}(t, s) \big|_{t=i-s} + r_{r}(t, s) \big|_{t=r-s} = 2M\bar{X}(0)\bar{X}'(0),
\]
whence, taking into account III, it follows that the distribution of the vector \( (X(t), X'(t)) \) is nondegenerate for \( t = 0 \) and thus, also in some \( \Delta \)-neighborhood of the point \( 0 \). Consequently, [3],
\[
\mathcal{M}_u = \int_{-\Delta}^{\Delta} \int_0^\infty y^2 P_X(t, t)(u, y) \, dy \, dt < \infty.
\]

The finiteness of the quantities \( \alpha_2(u) \) and \( \beta_2(u) \) is proved in Sec. 5.

Inequalities (1) are obvious and inequalities (2) follow from the relations
\[
P \left( \max_{|t| \leq \Delta} X(t) \geq u \right) = P \left( X(-\Delta) \geq u \right) + P \left( X(-\Delta) < u, \max_{|t| \leq \Delta} X(t) \geq u \right),
\]