VARIABILITY OF ALL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

OF ODD ORDER

T. A. Chanturiya

We consider the linear differential equation

\[ u^{(n)} = p(t) u, \quad (1) \]

where \( n \geq 3 \) is an odd number, \( p: \mathbb{R}^+ \to \mathbb{R} \) is a continuous function. By a solution of (1) will be understood a nontrivial solution defined in the interval \( \mathbb{R}^+ \). A solution of (1) is called variable if it has an even number of zeros, and nonvariable in the opposite case.

In the present paper we prove the following:

**THEOREM 1.** Let there exist sequences of segments \( \{[a_k, b_k]\}_{k=1}^\infty \) and numbers \( \{c_k\}_{k=1}^\infty \) such that

\begin{align*}
&b_{2k-1} < c_k < a_{2k}, \quad b_{2k} < a_{2k+1}, \\
p(t) > 0 \quad &\text{for } t \in [a_{2k-1}, c_k], \\
p(t) < 0 \quad &\text{for } t \in [c_k, b_{2k}], \\
&\lim_{k \to +\infty} (b_k - a_k)^n p_k = +\infty,
\end{align*}

\[ (2), (3), (4) \]

where \( p_k = \min \{\|p(t)\| : t \in [a_k, b_k]\} \). Then all solutions of (1) are variable.

First we give some definitions and auxiliary assertions.

We shall say that (1) has property A if each solution of this equation is either variable or satisfies the condition \( |u^{(i)}(t)| \downarrow 0 \) for \( t \uparrow +\infty \) \((i = 0, \ldots, n-1)\).*

We shall say that (1) has property B if each solution of this equation is either variable or satisfies the condition \( |u^{(i)}(t)| \uparrow +\infty \) as \( t \uparrow +\infty \) \((i = 0, \ldots, n-1)\).

We note that if \( p(t) \geq c > 0 \) \((p(t) \geq c > 0)\) for \( t \in \mathbb{R}^+ \), then (1) has property A (property B).

It is easy to see the validity of the following lemma.

**LEMMA 1.** Let \( p(t) \leq 0 \) for \( t \in \mathbb{R}^+ \) and (1) have property A. Then any solution of this equation is either variable or satisfies the condition

\[ (-1)^i u^{(i)}(t) u(t) > 0 \quad \text{for } t \in \mathbb{R}^+ \quad (i = 0, \ldots, n-1). \]

Let \( t_0 \in \mathbb{R}^+ \). In [1] there was introduced the set \( F_{up} \) such that \( t_1 \in F_{up} \) if \( t_1 > t_0 \) and there exists a solution \( u \) of (1) satisfying the conditions \( u(t_0) = 0, \ u(t) \geq 0 \) for \( t \in [t_0, t_1] \) and \( u^{(i)}(t_1) \geq 0 \) \((i = 0, \ldots, n-1)\), where equality holds for at least one \( i \). It was proved there also that if

\[ p(t) \geq 0 \quad \text{for } t \in \mathbb{R}^+ \]

and (1) has property B, then \( F_{up} \) is nonempty, bounded, and closed. We set \( \eta_{up} = \max F_{up} \).

We give two lemmas from [1], which we shall need in what follows. Let the function \( q: \mathbb{R}^+ \to \mathbb{R} \) be continuous and

\[ p(t) \geq q(t) \geq 0 \quad \text{for } t \in \mathbb{R}^+ \]

\[ (5), (6) \]

*The notation \( (t) \downarrow 0 \) as \( t \uparrow +\infty \) denotes that \( u \) is monotone decreasing, starting with some value of the argument, and tends to zero as \( t \to +\infty \). One should understand analogously the notation \( u(t) \uparrow +\infty \) as \( t \to +\infty \).

We consider the equation
\[ v^{(n)} = q(t) v. \] (7)

**Lemma 2.** Let (1) and (7) have property B and let (6) hold. Then for any \( t_0 \leq R_+ \) we have \( \eta_{\text{op}} \leq \eta_{\text{eq}} \).

**Lemma 3.** Let (1) have property B, let (5) hold, and let \( u \) be a solution of (1), vanishing at the point \( t_0 \). Then either there exists a point \( t_1 \leq [t_0, \eta_{\text{op}}] \) such that \( u(t_1) = 0 \), or
\[ u^{(i)}(\eta_{\text{op}}) u(\eta_{\text{op}}) \geq 0 \quad (i = 0, \ldots, n - 1). \]

It is easy to verify that if
\[ p(t) \equiv \text{const} = c > 0, \]
then for any \( t_0 \leq R_+ \)
\[ \eta_{\text{op}} = t_0 + \eta_*/\sqrt{c}, \]
where \( \eta_* = \eta_{\text{eq}} \) for \( q(t) \equiv 1 \).

**Lemma 4.** Suppose one has
\[ (b - a)^n p(t) \geq \eta_*^n \quad \text{for} \quad t \in [a, b]. \]
Then any solution of (1) either satisfies the condition
\[ u^{(i)}(b) u(b) > 0 \quad (i = 0, \ldots, n - 1) \]
or vanishes at some point of the interval \([a, b]\).

**Proof.** It will be assumed that
\[ p(t) = p(a) \quad \text{for} \quad t < a, \]
\[ p(t) = p(b) \quad \text{for} \quad t > b. \]
Let \( u \) be an arbitrary solution of (1). If we set
\[ u(b - t) = w(t) \quad \text{for} \quad t \in R_+, \]
then we see that \( w \) satisfies the equation
\[ w^{(n)} = p^*(t) w, \] (9)
where \( p^*(t) = -p(b - t) \) for \( t \in R_+ \).

It is obvious that
\[ -p^*(t) \geq \eta_{\text{eq}}^n/(b - a)^n \quad \text{for} \quad t \in R_+. \]
Hence (9) has property A. But then it follows from Lemma 1 that either \( w \) satisfies the condition
\[ (-1)^i w^{(i)}(t) w(t) > 0 \quad \text{for} \quad t \in R_+. \]
or \( w \) is variable, i.e., in view of (8), either
\[ u^{(i)}(t) u(t) > 0 \quad \text{for} \quad t \leq b \]
or \( u \) vanishes at some point of the interval \([a, b]\). In the second case let \( t_0 \) be the largest zero of the function \( u \) in the interval \([a, b]\). Let us assume that \( t_0 < a \) and we set
\[ q(t) = \eta_{\text{eq}}^n/(b - a)^n \quad \text{for} \quad t \in [t_0, +\infty]. \]
In view of the fact that \( p(t) \geq q(t) \) for \( t \in [t_0, +\infty] \), (1) and (7) have property B. Moreover, according to Lemma 2,
\[ \eta_{\text{op}} \leq \eta_{\text{eq}} = t_0 + \eta_*/\sqrt{q(t)} = t_0 + b - a < b. \]
But then according to Lemma 3
\[ u^{(i)}(\eta_{\text{op}}) u(\eta_{\text{op}}) \geq 0 \quad (i = 0, \ldots, n - 1), \]
and hence
\[ u^{(i)}(b) u(b) \geq 0 \quad (i = 0, \ldots, n - 1). \]