1. On the basis of an algorithm proposed by Schensted [2], Knuth [1] constructed bijections of the set of nonnegative integral matrices and the set of \((0, 1)\)-matrices onto suitable subsets of ordered pairs of generalized Young tableaux. Knuth also considered the restriction of the first of these bijections to the symmetric matrices with given trace, which gives rise to a mapping onto the following set of Young tableaux: by symmetry of the matrix, both the tableaux in the pair coincide, and the imposing of constraints on the trace is equivalent to selecting Young tableaux with certain diagrams only. Burge [3] modified the way in which Schensted's algorithm was used and constructed a series of bijective maps of different subsets of symmetric matrices, including \((0, 1)\)-matrices, onto subsets of generalized Young tableaux. In each of these subsets, the constraint imposed on the matrices leads to corresponding restrictions on the Young diagrams appearing in the mapping of generalized tableaux.

By interpreting integral nonnegative matrices as contiguity matrices \([4, 5]\) of some graph or multigraph of a certain class (depending on the constraints imposed), it is possible to consider the above results in the context of graph-theory. In this note we construct a bijective map of the set of \(k\)-vertex \(p\)-tournaments onto a set of generalized Young tableaux with a single well-determined Young diagram. A relation with group representation theory is noted. In Sec. 3 it is proved that the corresponding sets and subsets have the same cardinality. Section 4 is devoted to constructing the maps.

2. We define some concepts. A Young diagram (or Ferrer graph) \(\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]\) is the image in the plane of a partition \(n = \lambda_1 + \lambda_2 + \ldots + \lambda_n\), \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\) of a nonnegative integer. The squares in the Young diagram (or points of the Ferrer graph) are arranged in horizontal rows, which in turn are arranged one under the other so that the left-hand squares form a vertical column. The number of squares in the \(r\)-th row, counting from top to bottom, is equal to \(\lambda_r\). An example of a Young diagram is

\[
[\lambda] = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
\]

A generalized Young tableau with diagram \([\lambda]\) is constructed by placing one element in some linearly ordered finite set \(K\) in each square of the diagram, observing the following rules:

\[
\begin{align*}
\text{a)} & \quad \text{the numbers in the rows do not decrease from left to right;} \\
\text{b)} & \quad \text{the numbers in the columns increase from top to bottom.}
\end{align*}
\]
The ordered set $K$ is usually taken to be $K = \{1, 2, \ldots, k\}$ (this is the choice throughout the sequel). The perimeters of the squares may be omitted in a tableau. An example of a generalized Young tableau with the diagram (1) is

$$
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 2 & 3 & 4 \\
3 & 4 & 4 & 4
\end{array}
$$

(3)

For each $t \in K$ we write $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ for the Young diagram obtained by removing all squares with elements greater than $t$ in a generalized tableau. The Young diagrams $[\lambda]$ and $[\lambda]_{i-1}$ differ precisely by the squares in the diagram $[\lambda]$ containing the element $t$ in the generalized tableau, and they are in turn uniquely determined by the location of the element $t$ in the given tableau. In this notation, the conditions for constructing (2) are expressed by the system of inequalities

$$
\forall t \in K \quad \forall t > 1 \quad \forall t > 1 \quad (\lambda_i > \lambda_{i-1} \geq \lambda_i > \lambda_{i+1}).
$$

(4)

This sets up a bijective correspondence between the set of integral nonnegative matrices $|\lambda_{i,j}|$ satisfying (4) (called Gel'fand–Tseitlin diagrams [6]) and the set of generalized Young tableaux (with a fixed $K$). In the theory of group representations, both sets are used in uniquely indexing bases of irreducible representations of the full linear group $SL(k)$ [6-8].

3. Throughout the sequel the set $K$ will be taken as the set of vertices of $p$-tournaments. The contiguity matrix $A_0 = |a_{i,j}(0)|$ of a $p$-tournament $\theta$ satisfies the following conditions:

$$
\forall i \in K \quad (a_{i,i}(0) = 0),
\forall i \neq j \quad (a_{i,j}(0) + a_{j,i}(0)) = p,
$$

(5)

i.e., each pair of vertices $\{i, j\}$ in the $p$-tournament is joined by $p$ arcs: $a_{i,j}(0)$ of the arcs go from vertex $i$ to vertex $j$, and $a_{j,i}(0)$ go from $j$ to $i$; the arcs are not marked (they have the same color).

Let: 1) $x_1, x_2, \ldots, x_k$ be complex variables, 2) $X = |\delta_{i,j}|$ a diagonal matrix $(i, j \in K)$, 3) $b_{i,j}(0) = XA_0$ (product matrix), 4) $P$ the set of all $p$-tournaments with vertices $K$, 5) $C_{<d>}$ the number of $p$-tournaments with set of initial semidegrees $<d> = (d_1, d_2, \ldots, d_k)$. We have

$$
\sum_{i \in P} \Pi_{i,j \in K} b_{i,j}(0) = \sum_{<d>} C_{<d>} x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k} =
\Pi_{i,j \in K} \left( \sum_{i \in J} \Pi_{i,j \in P} \left( \frac{x_1^{d_1 - 1}}{x_1^{d_1 - 1}} \right) \right) = \frac{\Pi_{i,j \in K} (x_1^{d_1 - 1} - x_1^{d_2 - 1})}{\Pi_{i,j \in K} (x_1^{d_1 - 1} - x_1^{d_2 - 1})} = \frac{1}{x_1^{d_1 - 1}}.
$$

(6)

Here the first equality is a direct consequence of the definitions. The second is valid owing to the equivalence of summation over all tournaments $\theta \in P$ and independently choosing for each pair of vertices all possible $p + 1$ different orientations of the arcs joining the vertices. The last two equalities are from elementary algebra, although they reflect a fact of importance to us, viz., the generating function for the numbers $C_{<d>}$ is equal to the ratio of two Van der Monde determinants.

Obviously, $\forall <d> (\sum_{i=1}^k d_i = 1/2pk (k - 1) = n)$. In a tensor space of rank $n$ of the group $GL(k)$, finite-dimensional irreducible representations of $GL(k)$ given by Young diagrams $[\lambda]$ with $\sum_{i=1}^k \lambda_i = n$ are realized. If matrix $X' \in GL(k)$ is diagonalizable and $x_1, x_2, \ldots, x_k$ are the eigenvalues of $X'$, then the character of the corresponding representation is equal to [9]

$$
\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} = \frac{|x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k}|}{|x_1^{\lambda_1}|}.
$$

(7)