THE STRESSED STATE CAUSED BY A RESIDUAL STRAIN FIELD
IN PLATES WITH STRESS CONCENTRATORS

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We obtain an analytic solution of the problem of determining the stressed state caused by a given residual strain field (while taking account of three-dimensional effects) in a round plate with a concentric foreign inclusion. We study the influence of the geometric parameters of the given system, the nonlinearity of the distribution of the given residual strains over the thickness of the plate, and a possible jump in distortion at the surface of contact on the stressed state of the system. We discover an “internal boundary-layer” effect that is significant in the case when the given residual strain field is strongly gradient.

Problems of strength, reliability, and durability of equipment are closely connected with the problem of residual stresses in their elements. The most complete and reliable information possible on the stressed state of metal-glass joins caused by residual strains is also needed to develop effective methods of nondestructive control and testing of electrovacuum devices and their elements.

In the present article we use the sharpened plate theory [3], which makes it possible to determine all components of the stress tensor, to study a round plate with a concentric foreign inclusion in a stressed state caused by a given field of residual strains. We consider a round plate of radius \( r_1 \) containing a concentric foreign inclusion of radius \( r_0 \), assuming the plate and inclusion are both of thickness \( 2h \).

Determining the stressed state of this system in the sharpened formulation reduces to solving a system of differential equations [3]:

\[
\frac{d}{dr} [2\omega - (1 + a_1)L_0(\omega, \Phi)] = 0, \quad \frac{4}{15} h^4 a_2 \Delta^2 \omega - 2 \Delta \Phi + (1 + a_1)L_0(\omega, \Phi) = \frac{1}{3} a_3 h^2 \Delta \theta_2; \tag{1}
\]

\[
\Delta(\omega + \Phi_*) = 0, \quad \frac{d}{dr} \left[ \frac{4}{3} h^2 a_2 \Delta w + w + \Phi_* + 2 h a_3 \theta_1 \right] = 0; \tag{2}
\]

under the following coupling conditions (at \( r = r_0 \)):

\[
G^- L_1(\omega^-, \Phi^-) = G^+ L_1(\omega^+, \Phi^+); \quad \frac{d\Phi^-}{dr} = \frac{d\Phi^+}{dr}; \tag{3}
\]

\[
G^- L_2(\omega^-, \Phi^-) = G^+ L_2(\omega^+, \Phi^+), \quad G^- L_3(\omega^-, \Phi^-) = G^+ L_3(\omega^+, \Phi^+),
\]

\[
\omega^- = \omega^+; \quad \frac{d\omega^-}{dr} = \frac{d\omega^+}{dr}, \quad w^- = w^+; \quad \frac{d\Phi^-_*}{dr} = \frac{d\Phi^+_*}{dr}; \tag{4}
\]

\[
G^- \frac{d}{dr} (w^- + \Phi^-_*) = G^+ \frac{d}{dr} (w^+ + \Phi^+_*), \quad G^- L_4(w^-, \Phi^-_*) = G^+ L_4(w^+, \Phi^+_*). \tag{5}
\]

and the following boundary conditions (at \( r = r_1 \); the free boundary conditions):

\[
L_1(\omega^+, \Phi^+) = 0; \tag{6}
\]

\[
L_2(\omega^+, \Phi^+) = 0; \quad L_3(\omega^+, \Phi^+) = 0; \tag{7}
\]

\[
\frac{d}{dr} (w^+ + \Phi^+_*) = 0; \quad L_4(w^+, \Phi^+_*) = 0. \tag{8}
\]

Here and below the superscript “+” refers the quantity in question to the plate and the superscript “−” to the inclusion. In relations (1)–(8) \( G \) is the shear modulus, \( \nu \) is the Poisson coefficient, \( \Delta \) is the...
Laplacian, \( r \) is the radial coordinate, \( a_1 = (1 - 2\nu)^{-1} \), \( a_2 = \frac{1}{2}(1 - \nu)^{-1} \), \( a_3 = (1 - \nu)^{-1}(1 + \nu) \), and the \( L_i \) are differential operators in the unknown functions of the respective forms

\[
L_0(\omega, \Phi) = \omega + \Delta \Phi - a_3 \theta_0; \quad L_1(\omega, \Phi) = 2 \left( \frac{d^2 \Phi}{dr^2} - a_3 \theta_0 \right) - (1 - a_1)L_0(\omega, \Phi);
\]

\[
L_2(\omega, \Phi) = a_3 \theta_2 + \frac{2}{15} h^2 \left[ (1 - 2a_2) \omega - \frac{d^2 \omega}{dr^2} \right]; \quad L_3(\omega, \Phi) = \frac{d}{dr} \left( a_3 \theta_2 - \frac{4}{15} h^2 a_2 \Delta \omega \right);
\]

\[
L_4(\omega, \Phi) = \frac{d^2 \Phi}{dr^2} + (1 - 2a_2) \Delta (\omega - \Phi) - 6h^{-1} a_3 \theta_1.
\]

The quantities \( \theta_i \) are the integral characteristics of the residual strain field \( e_0 \):

\[
\theta_0 = \frac{1}{2} \int_{-1}^{1} e_0 \, d\zeta; \quad \theta_1 = \frac{1}{2} \int_{-1}^{1} e_0 \zeta \, d\zeta; \quad \theta_2 = \frac{1}{2} \int_{-1}^{1} e_0 (1 - 3\zeta^2) \, d\zeta; \quad \zeta = zh^{-1}.
\]

(9)

The residual strain field is determined by the experimental/theoretical method of [1]; for the plate it has the form

\[
e_0(\zeta, \xi) = A(p_0 + p_1 \zeta + p_2 \zeta^2)[1 + a_0^{-1} J_0(\gamma, \xi)]; \quad (10)
\]

\[
a_0 = 0.4028; \quad \gamma = 3.8317\zeta_1^{-1}; \quad \xi = r_\alpha^{-1}; \quad \xi_1 = r_\alpha^{-1}; \quad \alpha = \frac{1}{3} h^2(1 - \nu^2)^{-1}; \quad A \text{ is the amplitude and } J_0 \text{ is the Bessel function.}
\]

Certain distributions of residual strains over the height of this system correspond to the chosen relations among the \( p_i \). In the inclusion the residual strain field varies over the height according to the law (10) and is independent of the radius.

As can be seen from formulas (1)–(8), in general this problem divides into two separate problems: a symmetric problem (with respect to the plane \( z = 0 \)) (1), (3), (4), (6), (7), and a bending problem (asymmetric with respect to the plane \( z = 0 \)) (2), (5), (8).

**Solution of the bending problem.** We find the general solution of Eqs. (2) in the form

\[
w^- = \beta_1^+ \rho^2 [C_1^+ (1 - \ln \rho) - C_2^+ + C_3^+ - 2a_3 \theta_1^-] + C_4^- \ln \rho + C_5^-;
\]

\[
w^+ = \beta_1^+ \rho^2 [C_1^+(1 - \ln \rho) - C_2^+ + C_3^+ - \beta_2^+] + C_4^+ \ln \rho + C_5^+ + 4\lambda_1^{-2} a_0^{-1} \beta_1^+ \beta_2^+ J_0(\lambda_1 \rho);
\]

\[
\Phi^\pm = C_1^\pm \ln \rho + C_2^\pm + w^\pm.
\]

(11)

Here we have introduced the notation

\[
\rho = r_0^{-1} r; \quad \lambda_1 = 3.8317 \rho_1^{-1}; \quad \rho_1 = r_0^{-1} r; \quad \beta_1^\pm = 3(16h^2a^\pm)^{-1} r_0^2; \quad \beta_2^\pm = \frac{1}{3} h a_3^- Ap_1.
\]

The constants of integration \( C_i \) can be determined from conditions (5) and (8), as well as from the condition that the desired solution is bounded at the point \( r = 0 \).

On the basis of the solution (11) and the recovery formulas of [3] we obtain the following expressions for the bending components of the stresses in the inclusion and the plate respectively:

\[
\sigma^- = 2G^- h \zeta_1[(1 - 4a_0^-)2r_0^{-2}B_1^- - 3h^{-1}a_3^- \theta_1^-]; \quad \sigma^- = \sigma^+;
\]

\[
\sigma^+ = 2G^+ r_0^{-2} h \zeta_1((1 - 4a_0^+)^2B_1^+ + B_2^+ \rho^2 + 4\beta_1^+ \beta_2^+ a_0^{-1}[2a_0^+ J_0(\lambda_1 \rho) - (\lambda_1 \rho)^{-1} J_1(\lambda_1 \rho)] - 3h^{-1}r_0^2 a_3^+ \theta_1^+(\rho));
\]

\[
\sigma^+ = 2G^+ r_0^{-2} h \zeta_1((1 - 4a_0^+)^2B_1^+ + B_2^+ \rho^2 + 4\beta_1^+ \beta_2^+ a_0^{-1}[\lambda_1^{-1} J_1(\lambda_1 \rho) - (1 - 2a_0^+)^2 J_0(\lambda_1 \rho)] - 3h^{-1}r_0^2 a_3^+ \theta_1^+(\rho)).
\]

(12)

Here

\[
qB_1^- = 6h^{-1}[1 - (1 - 4a_0^+)\rho_1^2][a_3^- G^{-} \theta_1^- - a_3^+ G^+ \theta_1^+(r_0) + a_0^+ G^+ \beta_1^+ \beta_2^+ (1 - \rho_1^2) J_0(\lambda_1) + 2a_0^+ \rho_1^2 J_0(\lambda_1) - J_0(\lambda_1 \rho_1)]]; \quad \rho_1 = (1 - 4a_0^+) \rho_1^2 [a_3^- G^{-} \theta_1^- - a_3^+ G^+ \theta_1^+(r_0) + \frac{1}{2} \beta_1^+ \beta_2^+ (1 - 2a_0^+)^2 J_0(\lambda_1) - J_0(\lambda_1 \rho_1)]]
\]

\[
qB_1^+ = 6h^{-1}[a_3^- G^{-} \theta_1^- - a_3^+ G^+ \theta_1^+(r_0)] + \frac{1}{2} \beta_1^+ \beta_2^+ [2a_0^+ G^+ J_0(\lambda_1) - \rho_1^2 J_0(\lambda_1 \rho_1)] + G^- (1 - 4a_0^-)[2a_0^+ \rho_1^2 J_0(\lambda_1 \rho_1) - \lambda_1^{-1} J_1(\lambda_1)] - G^+ \lambda_1^{-1} J_1(\lambda_1)];
\]

\[
q = 4(1 - \rho_1^2)[(1 - 4a_0^-)G^- - (1 - 4a_0^+)G^+] + 16\rho_1^2 a_3^2 (1 - 4a_0^+)^2 G^+;
\]

\[
B_1^+ = -\frac{1}{2} \rho_1^2 [4(1 - 4a_0^+) B_1^+ + \beta_1^+ \beta_2^+ a_3^2 J_0(\lambda_1 \rho_1)].
\]