ON THE STRESSED STATE IN A FIBER-REINFORCED COMPOSITE MATERIAL WITH TWISTED FILAMENTS

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Using the model of a piecewise-homogeneous body in the framework of the linear theory of elasticity we study the distribution of internal stresses in a fiber-reinforced composite material with twisted filaments under loading "at infinity" by uniformly distributed normal strains in the direction of the filaments. It is assumed that the concentration of filaments is rather sparse and their interaction is not taken into account.

The author and A. N. Guz' [1] have used the model of a piecewise-homogeneous body in the framework of the linear theory of elasticity to propose an approach to the study of the stressed state in a fiber-reinforced composite material with distorted filaments. In doing so, they assumed that the concentration of filaments is rather sparse and did not take account of their interaction. Applying the approach of [1], they studied [2] the distribution of self-balancing normal and tangential stresses at the interface of a filament and the matrix in a fiber-reinforced composite material for the case when the midline of the filament is a planar sinusoidal curve. The study of the stress distribution in a fiber-reinforced composite material with twisted filaments is of great interest. Whitney [3] has studied the influence of twisting the filaments on the magnitude of the reduced mechanical characteristics in the case of compression along the filaments, but this approach does not give reliable information about the quantitative (and in some cases also qualitative) nature of the distribution of stresses in each of the components of the composite material being studied. This information can be obtained only in the context of the model of a piecewise-homogeneous body with the application of a strictly three-dimensional theory.

In the present article we study the distribution of stresses in a fiber-reinforced composite material with twisted filaments, following the approach of [1]. It is assumed in doing so that the concentration of filaments is rather sparse, and their interaction is not taken into account.

1. Formulation of the problem and method of solution. Consider an infinite elastic body reinforced with a single twisted filament. The materials of the filament and the matrix are assumed homogeneous and isotropic. We attach to the filament a Cartesian coordinate system \((x_1, x_2, x_3)\) and a cylindrical coordinate system \((r, \theta, x_3)\) (Fig. 1). We study the stressed state in this body under stretching at infinity with uniformly distributed normal forces \(\{\rho\}\) in the direction of the \(Ox_3\)-axis. We note that \(\{\rho\}\) is the stress averaged over the whole area of the composite under consideration, on which the normal force acts in the direction of the \(Ox_3\)-axis.

Quantities related to the filaments will be denoted by the superscript \((2)\) and those related to the matrix by the superscript \((1)\). Inside the filament and the matrix in the cylindrical coordinate system we write the equilibrium equations, Hooke's law, and the Cauchy relations:

\[
\frac{\partial \sigma^{(m)}_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma^{(m)}_{r\theta}}{\partial \theta} + \frac{\sigma^{(m)}_{33}}{r} + \frac{\sigma^{(m)}_{rr}}{r} - \frac{\sigma^{(m)}_{\theta\theta}}{r} = 0, \quad \ldots \quad (m = 1, 2);
\]

\[
\sigma^{(m)}_{rr} = \frac{E^{(m)}}{1 + \nu^{(m)}} \left( \frac{\nu^{(m)}}{1 - 2\nu^{(m)}} \varepsilon^{(m)}_{rr} + \varepsilon^{(m)}_{33} \right), \quad \ldots \quad ; \quad \varepsilon^{(m)}_{rr} = \frac{\partial u^{(m)}_{rr}}{\partial r}, \quad \ldots \quad . \tag{1}
\]

Here we have used standard notation.

We assume that on the interface of the filament and matrix complete coupling conditions hold. Denoting this surface by \(S\), we write these conditions as

\[
(\sigma^{(1)}_{rr} n_r + \sigma^{(1)}_{r\theta} n_\theta + \sigma^{(1)}_{\theta\theta} n_\theta) \big|_S = (\sigma^{(2)}_{rr} n_r + \sigma^{(2)}_{r\theta} n_\theta + \sigma^{(2)}_{\theta\theta} n_\theta) \big|_S.
\]
Fig. 1

\[\left(\sigma_{r\theta}^{(1)} n_r + \sigma_{\theta\theta}^{(1)} n_\theta + \sigma_{\theta3}^{(1)} n_3\right) |_S = \left(\sigma_{r\theta}^{(2)} n_r + \sigma_{\theta\theta}^{(2)} n_\theta + \sigma_{\theta3}^{(2)} n_3\right) |_S; \]

\[\left(\sigma_{r3}^{(1)} n_r + \sigma_{\theta3}^{(1)} n_\theta + \sigma_{33}^{(1)} n_3\right) |_S = \left(\sigma_{r3}^{(2)} n_r + \sigma_{\theta3}^{(2)} n_\theta + \sigma_{33}^{(2)} n_3\right) |_S; \]

\[u_r^{(1)} |_S = u_r^{(2)} |_S; \quad u_\theta^{(1)} |_S = u_\theta^{(2)} |_S; \quad u_3^{(1)} |_S = u_3^{(2)} |_S, \]

where \(n_r, n_\theta, \) and \(n_3\) are the components of the unit normal vector to the surface in the cylindrical coordinate system.

The middle line of a filament is assumed to be a helix (in Fig. 1 this curve is marked by the number 1). In the chosen cylindrical coordinate system we write the equation of this curve in the following parametric form:

\[r = L = \text{const}; \quad \theta = \frac{2\pi}{h} t; \quad x_3 = t, \quad (3)\]

where \(L\) is the winding radius of the helix and \(h\) is the winding step. We assume that \(L < h\) and introduce the dimensionless small parameter \(\varepsilon = L/h.\)

We shall assume, as in [1, 2], that the cross sections of a filament perpendicular to the middle curve of the filament are circles and that the radius of these circles \(R\) is constant along the whole length of the filament. Using (3) and the conditions on the cross section of a filament, we deduce the equations of the surface \(S\) in the cylindrical coordinate system in the following form:

\[r = \frac{1}{1 + 4\pi^2 \varepsilon^2 \sin^2(\alpha t - \theta)} \left\{ \varepsilon h \cos(\alpha t - \theta) + R \left[ 1 + \left( 4\pi^2 \varepsilon^2 - \frac{h^2 \varepsilon^2}{R^2} (1 + 4\pi^2 \varepsilon^2) \right) \sin^2(\alpha t - \theta) \right] \right\}; \]

\[x_3 = t + 2\pi \varepsilon r(t) \sin(\alpha t - \theta); \quad \alpha = \frac{2\pi}{h}. \quad (4)\]

From (4), after certain transformations, we obtain expressions for the unit vectors \(n_r, n_\theta, \) and \(n_3\) in the form

\[n_r = n_r(\theta,t); \quad n_\theta = n_\theta(\theta,t); \quad n_3 = n_3(\theta,t). \quad (5)\]

The explicit form of formula (5) is given in [1, 2].

The quantities that characterize the stress-strain state in the filament and in the matrix will be sought as series in the parameter \(\varepsilon\) in the following form:

\[\sigma_{rr}^{(m)} = \sum_{k=0}^{\infty} \varepsilon^k \sigma_{rr}^{(m)k}; \ldots; \quad \varepsilon_{rr}^{(m)} = \sum_{k=0}^{\infty} \varepsilon^k \varepsilon_{rr}^{(m)k}; \ldots; \quad u_r^{(m)} = \sum_{k=0}^{\infty} \varepsilon^k u_r^{(m)k}; \ldots \quad (6)\]

In determining the zeroth approximation we find that the size of this approximation corresponds to the stress-strain state of the body in question with a rectilinear filament and balanced external forces. Therefore