Series Solutions for Steady Unsaturated Flow in Irregular Porous Domains

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Abstract. For the most part, analytical solutions for steady unsaturated infiltration have been restricted to infinite and semi-infinite seepage geometries, using the quasi-linear approximation for the hydraulic conductivity. We provide analytical series methods to solve the steady quasi-linear flow equations, in finite irregular seepage geometries. Unlike the classical approach, the series method has been modified, to cater for arbitrary boundary geometry and surface recharge distributions. The matrix flux potential and the stream function both satisfy the same governing partial differential equation, and the stream function formulation is used to estimate the series coefficients. For a finite vadose zone, the stream function solution does not uniquely determine the matrix flux potential, when flux boundary conditions are used. Consequently, the stream function solution applies to a range of moisture distributions, for given infiltration and evapotranspiration rates through the surface.

Key words: unsaturated flow, series solutions, quasi-linear method, irregular domains.

1. Introduction

Steady infiltration through a porous medium will lead to a saturated region, for a sufficiently high recharge rate. However, at the lower recharge rates applicable to some arid regions, the seepage domain will remain unsaturated. Consequently, predictions of the steady water concentration field in unsaturated porous domains are necessary, to quantify the water and solute transport processes. A key feature of unsaturated flow is the highly nonlinear relationship between the moisture content and the hydraulic conductivity. However, for steady infiltration, the relevant nonlinear boundary value problem can be made more tractable by invoking the quasi-linear approximation. Thus, the governing partial differential equation is reduced to linear form by a nonlinear transformation, while still maintaining the essential nonlinear character of the transport process [4, 8, 9].

The efficacy of the quasi-linear approximation has been reviewed by Pullan [13]. In this approximation, the soil hydraulic conductivity is expressed as an exponential of the moisture potential. After applying the Kirchhoff transformation, the dependent variable is the matrix flux potential, which then obeys a linear elliptic partial differential equation. This allows exact solutions to be obtained in
one dimension [4] and in higher dimensions [8]. Alternatively, the problem can be formulated using a stream function [14], which can lead to simplifications in the solution process, when derivative boundary conditions apply.

Analytical solutions can be obtained for the two-dimensional quasi-linear flow equation, by using the method of separation of variables. Solutions have been obtained for infinite [8, 12] or semi-infinite porous domains [1, 7, 10, 11, 14, 17], and a rectangular strip of finite depth [11]. In practice, however, vadose zones are of finite depth and length, with irregular surface and basal geometries. We consider a porous permeable layer, overlaying an impermeable base, with the only permeable boundary at the soil surface. The extremities of the porous region will consist of impermeable, vertical dykes, or the intersection of the underlying impermeable base and the soil surface. Unfortunately, the classical series approach has only been applicable to separable problems with regular (i.e., rectangular, circular, etc.) boundary geometries (e.g., [3]). In a new analytical approach introduced here, we incorporate an arbitrary boundary geometry by expanding the solution in terms of a suitable nonorthogonal basis.

Recently, the classical series method has been modified to cater for steady saturated seepage problems, represented by Laplace’s equation with arbitrary boundary geometries [16]. The series expansion for the hydraulic head provides an analytic solution throughout the entire flow domain, for comparatively little computational effort. Although saturated seepage is governed by Laplace’s equation, the method appears to be applicable to any separable partial differential equation, including the Helmholtz equation whose applications include steady unsaturated flow [12].

In this paper, the series method is applied to the infiltration problem, for finite two dimensional porous domains with arbitrary shape. The problem is formulated in terms of the matrix flux potential and the stream function, using derivative boundary conditions. An elementary analysis reveals that the matrix flux potential is not determined uniquely by the stream function solution, when the rate of infiltration/evapotranspiration is specified along the soil surface. This fact is examined, and we provide bounds on the range of allowable solutions for the matrix flux potential. We also derive flow solutions for a variety of boundary geometries.

2. Mathematical Problem Description

We consider steady two-dimensional unsaturated flows described by the Richards equation:

\[
\frac{\partial \theta}{\partial t} = 0 = \nabla_\ast \cdot (K_\ast \nabla_\ast \Psi_\ast) + \frac{\partial K_\ast}{\partial \zeta_\ast},
\]

where \( \theta \) is the volumetric water content, \( \nabla_\ast = (\partial/\partial x_\ast, \partial/\partial z_\ast) \) is the grad operator, \( z_\ast \) is the vertical coordinate (increasing upwards) and \( K_\ast \) is the hydraulic conductivity, which is a function of the moisture potential \( \Psi_\ast \).