ON UPPER BOUNDS OF FOURIER-WALSH COEFFICIENTS

A. V. Efimov

An upper bound is established for the upper bounds of the Fourier-Walsh coefficients $a_n(f)$ whose modulus of continuity $\omega(\delta, f)$ does not exceed a given modulus of continuity $\omega(\delta)$. In the case of convex majorants of $\omega(\delta)$, these bounds are attained for individual ordinal numbers $n$.

Let $\omega(\delta)$ be a given modulus of continuity, i.e., a function, continuous for $\delta \equiv 0$, and satisfying the conditions (see, [1])

$$\omega(0) = 0 \text{ and } \omega(\delta_1) - \omega(\delta_2) \leq \omega(\delta_2 - \delta_1),$$

$$0 < \delta_1 < \delta_2.$$

We denote by $KH[\omega]$ (by $H[\omega]$ when $K = 1$) the class of continuous functions $f(x)$, given on $[0, 1]$, whose moduli of continuity

$$\omega(\delta, f) = \sup_{|h| \leq \delta, x \in \mathbb{R}, x+h \leq t} \max |f(x+h) - f(x)|, 0 < \delta < 1,$$

do not exceed a given modulus of continuity, i.e.,

$$\omega(\delta, f) \leq K \omega(\delta).$$

The class of functions $f(x) \in KH[\omega]$ with period 1 we shall denote by $KH^*[\omega]$ and, when $K = 1$, by $H^*[\omega]$. Furthermore, let $\{W_n(x)\}$ be the system of Walsh functions orthonormalized on $[0, 1]$. With this, we take the numbering system in the system in the sense of Paley (see, [2]), i.e., if

$$r_n(x) = \text{sign} \sin 2^n \pi x, \quad n = 0, 1, 2...$$

are Rademacher functions, and if

$$k = 2^n + 2^m + ... + 2^m, \quad k_1 > k_2 > \ldots > k_m > 0,$$

then

$$W_k(x) = \prod_{j=1}^m r_{k_j}(x).$$

We denote by $a_k = a_k(f)$ the Fourier-Walsh coefficients, i.e.,

$$a_k(f) = \sum_{n=0}^t f(x) W_k(x) dx.$$

We shall study the behavior of the quantities

$$A_k[\omega] = \sup_{f \in H[\omega]} |a_k(f)|.$$

The analogous problems for the upper bounds of the Fourier coefficients of $2\pi$-periodic functions for convex moduli of continuity $\omega(\delta)$, i.e., such that...
were considered by Lebesgue [3] and, for arbitrary moduli of continuity \( \omega(\theta) \), by the author in [4, 5]. It was shown (see, [4]) that, in this case,

\[
A_k[\omega] = \sup_{f \in H[\omega]} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos kn}{\sin kn} \int f(x) \cos kx \, dx
\]

where \( 2/3 < \theta_k \leq 1 \), with \( \theta_k = 1 \) for convex moduli of continuity \( \omega(\delta) \). It was also shown ([1, 4, 5, 6, 7] see, Refernativyi Zhurnal, Matematika, 1968, 4B 151 and others) that the constants \( A_k[\omega] \) enter into a host of constants in the theory of function approximation.

In the present note we shall give an upper bound for the quantities \( A_k[\omega] \) and prove that, for certain ordinal numbers, this upper bound is attained. Namely, we shall prove the

**THEOREM.** For all \( n = 2^n_1 + 2^n_2 + \ldots + 2^n_m, n_1 > n_2 > \ldots > n_m \geq 0 \) we have the bound

\[
A_n[\omega] \leq 2^{n_1/2^n_1+1} \int_0^{\infty} \omega(2t) \, dt,
\]

where, in case the moduli of continuity \( \omega(\delta) \) are convex, the bound in (2) is attained when \( m = 2, n_2 = n_1 - 1, \) i.e.,

\[
A_{2n_12n_1-1}[\omega] = 2^{n_1/2^n_1+1} \int_0^{\infty} \omega(2t) \, dt.
\]

If, however, \( m = 1 \), then

\[
A_{2n_1}[\omega] \leq 2^{n_1/2^n_1+1} \int_0^{\infty} \omega(2t) \, dt + O\left(\int_0^{1/2^n_1+1} \omega(t) \, dt\right) + \int_0^{(1/2^n_1+1)} \frac{f(t) - f(1/2^n_1)}{2^n_1+1} \, dt,
\]

and, for any class of periodic functions, we have

\[
A_{2n_1}[\omega] \leq 2^{n_1/2^n_1+1} \int_0^{\infty} \omega(2t) \, dt.
\]

In case the moduli of continuity \( \omega(\delta) \) are convex, we then have the equality sign in (4) and (5), and the following relationship holds

\[
A_{2n_1}[\omega] = A_{2n_12n_1-1}[\omega] \quad (\omega(\delta) \neq 0).
\]

**Proof.** We first prove the bound of (4). Since functions \( W_{2n_1} \) are odd with respect to the point \( x = (2v + 1)/2^{n_1+1} \), where, for all \( k = 0, 1, 2, \ldots, n_1 \),

\[
r_{2n_1}(2v + 1)/2^{k+1} - t = -r_{2n_1}(2v + 1)/2^{k+1} + t = 1
\]

when \( 0 < t < 1/2^{k+1} \), we then have

\[
a_{2n_1}(f) = \sum_{t=0}^{1/2^n_1} f(x) r_{2n_1}(x) \, dx = \sum_{t=0}^{1/2^n_1} f(1/2 - t) r_{2n_1}(1/2 - t) \, dt
\]

where

\[
\begin{align*}
\int_0^{1/2} f(1/2 + t) \, r_{2n_1}(1/2 + t) \, dt &= \int_0^{1/2} f(1/2 - t) \, r_{2n_1}(1/2 - t) \, dt \\
&= \int_0^{1/2} (f(1/2 + t) - f(3/4 - t)) \, r_{2n_1}(1/4 + t) \, dt \\
&+ \int_0^{1/2} (f(1/4 - t) - f(3/4 + t)) \, r_{2n_1}(1/4 - t) \, dt
\end{align*}
\]

\[
= \int_0^{1/4} [(f(1/4 - t) - f(1/4 + t)) + f(3/4 - t) - f(3/4 + t)] \, r_{2n_1}(1/4 - t) \, dt.
\]

\[\text{Note added in proof. For Walsh systems in the enumeration of Kaczmarz, relationships analogous to (2), (3), and (5) were established by methods different from ours by N. P. Khoroshko (Doctoral Dissertation, Dnepropetrovsk, 1969).} \]