We thus get that the inclusion \( \sigma_3(A) \subset \sigma_1(A) \) is strict. Using the fact that \( r(A^*) = r(A) \) in \( E_n = \mathbb{K} = \mathbb{C} \), we obtain analogously that the inclusion \( \sigma_3(A) \subset \sigma_0(A) \) is strict in the case \( 1/2 > \beta > 0 \), also.

LITERATURE CITED


CLOSED CLASSES OF k-VALUED LOGIC, ALL OF WHOSE CONGRUENCES ARE TRIVIAL

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In this article a criterion for the existence of only trivial congruences on a closed class of k-valued logic containing selectors is formulated and proved. All homomorphic, but nonisomorphic, images of a given closed class of k-valued logic containing selectors are described.

0. We know that the equality relation and the identically true relation are congruences on every universal algebra. Besides these congruences, the Post algebra of k-valued logic and every subalgebra of it have the arity congruence, "identifying" those and only those functions of the algebra which have the same arity. In the sequel these three congruences will be called trivial. Let us observe that except some degenerate cases, the trivial congruences are pairwise different on the subalgebras of the Post algebra of k-valued logic. The relation between the homomorphisms of an algebra and its congruences allows us to observe that all the homomorphisms of an algebra having only trivial congruences, different from isomorphisms, are trivial.

It has been shown in [1] that the Post algebra of k-valued logic and certain subalgebras of it have only trivial congruences. In [2] have been found for \( k = 2 \) all congruences on every subalgebra of the Post algebra not satisfying the condition of [1]. It follows from these results that, except a finite number of subalgebras, all subalgebras of the Post algebra for \( k = 2 \) have only trivial congruences. Let us observe that for \( k \geq 3 \) the Post subalgebras can have a continuum of congruences, a countable set of congruences, and also exactly \( n \) congruences, \( n \geq 3 \). Some of these subalgebras have been indicated in [2]. There exist similar examples also among subalgebras containing selectors.

Here we will consider subalgebras of the Post algebra of k-valued logic containing selectors. We take the operation of superposition [3] as the basis, in terms of which all the operations of [1] are expressed. The subalgebras of the Post algebra of k-valued logic are closed classes of k-valued logic [3]; both these terms will be used in the sequel.

In this article will be given necessary and sufficient conditions under which a closed class of k-valued logic containing selectors has only trivial congruences. The process of
construction of all congruences on an arbitrary subalgebra containing selectors is described. A classification of congruences and homomorphisms of subalgebras containing selectors is obtained. All the notions and the notation, used in this article without definition, are taken from [1, 3].

1. Let $E_k = \{0, 1, \ldots, k - 1\}$, $P_k^n$ be the set of all functions of $k$-valued logic depending on $n$ variables, $n \geq 1$, and $P_k = \bigcup_{n=1}^{\infty} P_k^n$ be the set of all functions of $k$-valued logic. The functions $g_i^n (x_1, \ldots, x_n) = x_i$, $1 \leq i \leq n$, $n = 1, 2, \ldots$, are called selectors or identities [1, 3]. In [1] certain operations were introduced on $P_k$ and it was suggested to consider $P_k$ as the algebra $\mathbb{R}_k$ with a signature from these operations. It has also been observed in [1] that the operation of superposition [3] can be expressed in terms of these operations. It is easily seen that the converse is also valid. Therefore, in the sequel we will use the operation of superposition, which is more convenient for subalgebras containing selectors.

Let us denote the algebra of all selectors by $S$. An equivalence relation on an algebra $\mathbb{A}$, $\mathbb{A} \subseteq \mathbb{R}_k$, which is stable with respect to all the operations is called a congruence on the algebra $\mathbb{A}$ [4]. Following [1], we denote by $\pi_0$, $\pi_1$, and $\pi_n$ the trivial congruences defined by the equality relation, the identically true relation, and the arity relation respectively. It has been shown in [2] that if an algebra $\mathbb{A}$ contains $S$, then only one of the relations $\pi = \pi_n$ and $\pi \subseteq \pi_n$ is valid for every congruence $\pi$ on $\mathbb{A}$. The latter means that $f \equiv g(\pi)$ implies $f \equiv g(\pi_n)$. Therefore, in the sequel we will consider only those congruences $\pi$ for which $\pi_n \supseteq \pi$. Let $\mathbb{A}, \mathbb{A} \supseteq S$, be a subalgebra of $\mathbb{R}_k$ and $\mathbb{A}_m = \mathbb{A} \cap P_k^m$, $m = 1, 2, \ldots$. If $\pi$ is a relation on the algebra $\mathbb{A}$, then we denote by $\pi^m$ the relation induced on $\mathbb{A}_m$ by $\pi$, $m > 1$. Let us give the following definition, which is important for the sequel.

Definition. An equivalence relation $\pi^m$ on $\mathbb{A}$ is said to be a compatible equivalence relation on $\mathbb{A}$ if the following conditions are satisfied:

1) For arbitrary natural number $m$ and function $f(x_1, \ldots, x_m)$ of $\mathbb{A}_m$ the relations $u_1 \equiv v_1(\pi^m)$, $u_1 \equiv v_1(\pi^m)$, imply that $f(u_1(x_1, \ldots, x_m), \ldots, u_m(x_1, \ldots, x_m)) = g(v_1(x_1, \ldots, x_m), \ldots, v_m(x_1, \ldots, x_m)) (\pi^m)$;

2) For arbitrary functions $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$ of $\mathbb{A}_n$, such that $u \equiv v(\pi^m)$, it follows from $u_1 \equiv v_1(\pi^m)$, $i = 1, 2, \ldots, n$, $u_i, v_i \equiv \mathbb{A}_n$ that $u(u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)) = v(v_1(x_1, \ldots, x_n), \ldots, v_n(x_1, \ldots, x_n)) (\pi^m)$.

The following proposition follows immediately from this definition.

Proposition 1. The equality relation and the identically true relation on $\mathbb{A}$ are compatible equivalence relations on $\mathbb{A}$ for every $n \geq 1$.

The following proposition is valid.

Proposition 2. Every congruence of an algebra $\mathbb{A}$ defines a compatible equivalence relation $\pi^m$ on $\mathbb{A}$.

We will prove the validity of the following lemma, which is the converse of Proposition 2.

Lemma 1. Let $\pi^m$ be a compatible equivalence relation on $\mathbb{A}$. Then there exists a congruence $\pi$ on $\mathbb{A}$ such that $\pi^m = \pi^m$.

Proof. Let us define the following relation $[\pi^m]$ on the algebra $\mathbb{A}$ by means of the compatible equivalence relation $\pi^m$:

$$f(x_1, \ldots, x_p) \equiv g(x_1, \ldots, x_q) ([\pi^m])$$

if and only if $p = q$ and the following relations holds for arbitrary functions $u_i$ and $v_i$ of $\mathbb{A}$, such that $u_i \equiv v_i(\pi^m)$, $i = 1, 2, \ldots, p$:

$$f(u_1(x_1, \ldots, x_n), \ldots, u_p(x_1, \ldots, x_n)) \equiv g(v_1(x_1, \ldots, x_m), \ldots, v_p(x_1, \ldots, x_m)) (\pi^m).$$

We now show that $\pi, \pi = [\pi^m]$, is the desired congruence. $[\pi^m]$ is an equivalence relation on the algebra $\mathbb{A}$: this follows easily from the definition $[\pi^m]$ and the fact that $\pi^m$ is an equivalence relation. Since $\mathbb{A} \supseteq S$ it is sufficient for proving the invariance of the relation $[\pi^m]$ under the operation of superposition to show that if the relations