The problem is considered of orthogonalization of J-symmetric representations of C*-algebras in the Pontryagin spaces \( \Pi_n \). It is proved that in spaces with finite rank of indefiniteness, every such representation is similar to a *-representation in a Hilbert space. Necessary and sufficient conditions are established for the existence of an invariant dual pair of subspaces for a J-symmetric operator algebra.

1. Kadison [1] formulated the following problem (the orthogonalization problem): is every representation of a C*-algebra in a Hilbert space similar to a *-representation. The recent papers [2-4] contain interesting results on the orthogonalization of nonsymmetric representations of C*-algebras of certain special classes, but in the general case, the answer is unknown. We consider here the problem of orthogonalization of J-symmetric representations of C*-algebras in spaces of the type \( \Pi_n \). The main result is the following: for \( n < \infty \) every J-symmetric representation of a C*-algebra is similar to a *-representation.

As is known (see [5]), every commutative C*-equivalent J-symmetric algebra possesses a dual pair of invariant subspaces. S. Ota [6] observed that C*-equivalence is also a necessary condition in the general case (of noncommutative J-symmetric algebras) for the existence of an invariant dual pair, and he posed the problem of the sufficiency of this condition. Using results on J-representations, we prove here the following assertions:

a) For \( n \geq \infty \), a J-symmetric algebra in \( \Pi_n \) possesses an invariant dual pair if and only if it is C*-equivalent;

b) the existence problem of a dual pair for C*-equivalent algebras in \( \Pi_n \) is equivalent to the general orthogonalization problem.

2. For any Banach space \( \mathcal{E} \), we shall let \( B(\mathcal{E}) \) denote the algebra of all bounded linear operators in \( \mathcal{E} \). We consider in \( B(\mathcal{E}) \) the topologies of uniform (u), strong (s), weak (w), and bounded-strong (bs) convergence. For any \( c > 0 \) we let \( h(\mathcal{E}, c) \) be the set of all operators \( A \in B(\mathcal{E}) \), such that \( \| \exp tA \| \leq c \) for all \( t \in \mathbb{R} \).

We shall need the following auxiliary result.

**Lemma 1.** Let the net \( \{A_\lambda\} \) bs-converge to the operator \( A \in B(\mathcal{E}) \), \( M = \sup \| A_\lambda \| \), and let \( f(z) \) be a function holomorphic in a neighborhood of the disk \( |z| \leq M \). Then \( f(A_\lambda) \xrightarrow{\text{bs}} f(A) \).

**Proof.** Using the fundamental theorems on the bs-topology (see [7]), it is not hard to prove that \( A_\lambda \xrightarrow{n} A^n (n \in \mathbb{N}) \). Hence \( p(A_\lambda) \xrightarrow{n} p(A) \) for any polynomial \( p \). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( |z| \leq M \). Fix \( \varepsilon > 0 \); since \( \sum_{n=0}^{N} |a_n| \cdot M^n \leq \infty \), we have \( \sum_{n=N+1}^{\infty} |a_n| \cdot M^n \leq \varepsilon \) for some \( N \in \mathbb{N} \). Put \( p(z) = \sum_{n=0}^{N} a_n z^n \); so that \( \| p(A) - f(A) \| \leq \varepsilon \) and \( \| p(A_\lambda) - f(A_\lambda) \| \leq \varepsilon \). Consequently, for any \( x \in \mathcal{E} \)

\[
\| (f(A_\lambda) - f(A)) x \| \leq 2\varepsilon \| x \| + \| (p(A_\lambda) - p(A)) x \|
\]

and hence \( \lim \| (f(A_\lambda) - f(A)) x \| \leq 2\varepsilon \| x \| \), which by the arbitrariness of \( \varepsilon \) and \( x \) means that \( f(A_\lambda) \xrightarrow{\text{bs}} f(A) \).

Proposition 1. For any Banach space $\mathcal{E}$ and any $c > 0$, the set $h(\mathcal{E}, c)$ is bs-closed in $B(\mathcal{E})$.

Proof. Let $\{A_t\}$ be a bounded net in $h(\mathcal{E}, c)$, converging strongly to $A \in B(\mathcal{E})$. By Lemma 1, $\exp itA_t \rightarrow \exp itA (\in \mathbb{R})$, which means that $\|\exp itA\| \leq \lim \|\exp itA_t\| \leq c$. Since $t$ is arbitrary, $A \in h(\mathcal{E}, c)$ and the proposition is proved.

By a representation of a Banach algebra $\mathcal{A}$ in a Banach space $\mathcal{E}$, we shall mean a continuous homomorphism of $\mathcal{A}$ into $B(\mathcal{E})$. Let $\mathcal{A}$ be an involutive Banach algebra, and assume that some involution $\varphi (A \mapsto A^\ast, A \in B(\mathcal{E}))$ is given in $B(\mathcal{E})$; if the representation is an involutive homomorphism of the algebra $\mathcal{A}$ into $(B(\mathcal{E}), \varphi)$, it is called a $\varphi$-representation.

A Banach $*$-algebra is said to be C*-equivalent if it becomes a C*-algebra if renormed by a topological equivalent norm. It is not hard to prove that a Banach $*$-algebra is C*-equivalent if and only if it is the image of a C*-algebra under a continuous $*$-homomorphism (it suffices to make use of a theorem on factor-algebras of C*-algebras and the minimality of the C*-norm, see [8, pp. 18, 29]).

Proposition 2. Let $\mathcal{E}$ be a Banach space, $\varphi$ a w-continuous involution in $B(\mathcal{E})$, $\pi$ a $\varphi$-representation of a C*-algebra $\mathcal{A}$ in $\mathcal{E}$. Then the $*$-algebra $(\pi(\mathcal{A})^\ast, \varphi)$ is C*-equivalent.

Proof. For any $*$-algebra $\mathcal{A}$, put $\text{Re} (\mathcal{A}) = \{a \in \mathcal{A}: a^\ast = a\}$. As was shown by Palmer [9], a necessary and sufficient condition for the C*-equivalence of a Banach $*$-algebra $\mathcal{A}$ is that there exist a $c > 0$ such that $\|\exp ia\| \leq c$ for all $a \in \text{Re} (\mathcal{A})$. It therefore suffices to prove the inclusion $\text{Re} (\pi(\mathcal{A})^\ast) \subseteq h(\mathcal{E}, c)$ for some $c > 0$. Since the involution $\varphi$ is w-continuous, we have $\text{Re} (\pi(\mathcal{A})^\ast) = \pi (\text{Re} (\mathcal{A})^\ast) = \pi (\text{Re} (\mathcal{A}))^\ast$ and for any $A \in \text{Re} (\pi(\mathcal{A})^\ast)$ there exists a net $\{a_t\} \subseteq \text{Re} (\mathcal{A})$ such that $\varphi(a_t) \rightarrow A$. For every $\varepsilon > 0$, the net $\{\pi(a_t)(1 + \varepsilon \pi(a_t))^{-1}\}$ is bounded, hence so is the net $\{\pi(a_t)(1 + \varepsilon\pi(a_t))^{-1}\}$. It is not hard to prove that for $\varepsilon < \|A\|^{-2}$

$$\pi(a_t)(1 + \varepsilon\pi(a_t))^{-1} \rightarrow A (1 + \varepsilon A)^{-1}$$

it suffices to observe that $(1 + \varepsilon\pi(a_t))^{-1} \leq (1 + \varepsilon\varphi(a_t))^{-1}$ and $\pi(a_t)^{-\ast} A$. Hence $A (1 + \varepsilon A)^{-1} \leq \pi (\text{Re} (\mathcal{A}))^\ast$, whence we obtain upon letting $\varepsilon$ tend to zero: $A \in \pi (\text{Re} (\mathcal{A}))^\ast$, i.e.,

$$\text{Re} (\pi(\mathcal{A})^\ast) \subseteq \pi (\text{Re} (\mathcal{A}))^\ast.$$ 

If further $a \in \text{Re} (\mathcal{A})$, then $\|\exp ia\| = \|\pi (\exp ia)\| \leq \|\pi\|$, which means that $\pi (\text{Re} (\mathcal{A})) \subseteq h(\mathcal{E}, \|\pi\|)$. Applying Proposition 1, we obtain: $\text{Re} (\pi(\mathcal{A})^\ast) \subseteq h(\mathcal{E}, \|\pi\|)^\ast = h(\mathcal{E}, \|\pi\|)$. The proposition is proved.

3. Let $H$ be a Hilbert space. As usual, we denote the standard involution in $B(H)$ taking an operator to its adjoint by $A \mapsto A^\ast$. Each unitary Hermitian operator $J \in B(H)$ further determines an involution

$$A \mapsto A^J = JA^\ast J \quad (A \in B(H))$$
in $B(H)$. In accordance with the notation introduced above, $*$-representations and J-representations of involutive algebras are representations symmetric with respect to the standard involution and J-involution in $B(H)$, respectively.

The choice of the operator $J$ also determines a new (indefinite) scalar product in $H$: $[x, y] = (Jx, y)$. Clearly, $[Ax, y] = [x, A^Jy]$ for $A \in B(H)$. Let $J = P_+ - P_-$ be the spectral decomposition, $x = \min \{\dim P_+H, \dim P_-H\}$. The space $(H, x, y \mapsto [x, y])$ is called a space of type $\Pi_x$.

We shall only need elementary results on indefinite metric spaces (see [10]). Vectors $x$ and $y$ in $H$ are said to be mutually $J$-orthogonal if $[x, y] = 0$. A space $L \subseteq H$ is said to be nondegenerate if its intersection with its $J$-orthogonal complement is trivial; otherwise, it is said to be degenerate. If a subspace is contained in its $J$-orthogonal complement, then it is said to be a neutral space. A subspace $L$ is said to be $J$-positive ($J$-negative) if $[x, x] > 0$ (respectively, $[x, x] < 0$) for $x \in L, x \neq 0$. In either of these two cases, $L$ is said to be sign-definite.

An operator $A \in B(H)$ is said to be $J$-positive if $[Ax, x] > 0$ for all $x \in H$.

The set of invariant subspaces of a family $\mathcal{M} \subseteq B(H)$ is denoted by $\text{lat} \mathcal{M}$. If $\mathcal{M}$ consists of nondegenerate subspaces, $\mathcal{M}$ is said to be nondegenerate. A representation of an algebra in $\Pi_x$ is called nondegenerate if its image is a nondegenerate algebra. The restriction of a representation $\pi: A \rightarrow B(H)$ to a subspace $L \subseteq \text{lat} \pi(\mathcal{A})$ is denoted by $\pi|L$. 

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