ASYMPTOTIC ESTIMATES OF INTEGRAL FUNCTIONS
DEFINED BY CANONICAL PRODUCTS

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The principal part of an asymptotic expansion at infinity of the logarithm of integral functions of finite order with simple positive zeros is determined. The asymptotic form is obtained with the aid of a Cauchy-type integral with smooth density.

General methods of obtaining asymptotic estimates of integral functions of finite order with an arbitrary (generally speaking) distribution of the roots are outlined in the well-known monographs [1, 2]. In the present paper, we study the asymptotic behavior of integral functions, all of whose zeros lie on a half-line. Asymptotic expansions of some integral functions with simple positive roots (the logarithm of a function is taken to within $O(1)$ for $z \to \infty$) are presented. Such asymptotic expansions are essential for solving certain boundary value problems in the theory of analytic functions.

§1. LEMMA 1. If the functional $y(x)$ is continuous on $[a, +\infty)$ ($a > 0$) and $y(x)$ is continuously differentiable for sufficiently large $x(x \geq b \geq a)$, while $y'(x)$ is monotonic and $|y'(x)| \geq k > 0$, then

$$
\omega(r) = \int_0^r \left( y(x) - \frac{1}{2} \right) \frac{dx}{x} = O(1) \quad (r \to \infty).
$$

Proof. By introducing the notation

$$
v(x) = \{y(x)\} - \frac{1}{2},
$$

and postulating (for $0 < a \leq b \leq r < \infty$),

$$
\omega(x) = \int_0^b \frac{v(x)}{x} \, dx + \int_0^r \frac{v(x)}{x} \, dx = \omega_1(b) + \omega_2(r),
$$

we get

$$
|\omega_1(b)| \leq \frac{1}{2} \ln \frac{b}{a}.
$$

Since $v(x)$ is integrable, $y'(x)$ is continuous, and $y(x)$ is strictly monotone on $[b, \infty)$, the following estimate holds for any (finite) $\lambda \geq b$ and $\mu \geq b$:

$$
\left| \int_\lambda^\mu v(x) y'(x) \, dx \right| = \left| \int_\lambda^\mu \{u\} \left( \frac{1}{2} \right) \, du \right| \leq \frac{1}{8}.
$$

Taking into consideration that $xy'(x)$ retains its sign for $x \geq b$, with the aid of the mean value theorem and (4), we obtain

$\{y(x)\}$ denotes the fractional part of $y(x)$.

\[ |\omega(r)| \leq \left| \int_0^r x^\alpha y'(x)\, dx \right| \leq \frac{4}{3} \max \left\{ \frac{1}{x|y'(x)|}, \frac{1}{\gamma|y'(x)|} \right\} . \]  

From (3), (5), and the condition \(|xy'(x)| \geq k > 0\), (1) follows. The lemma is proved.

It is noteworthy that the case where \(y(x) = \log x\) (and therewith \(xy'(x) = 1\)) has been analyzed in detail in [3], p. 318. An example of a function that does not satisfy the conditions of the lemma is the following: \(\log^\beta x, 0 < \beta < 1\).

§ 2. Let us now evaluate a Cauchy-type integral with a fluctuating density.

**Lemma 2.** According to the assumptions of Lemma 1 for \(z \to \infty, 0 < \varepsilon \leq \arg z \leq 2\pi - \varepsilon\)

\[ \Phi(z) = \int_0^\infty \frac{y(z)}{x(z - \varepsilon)}\, dx = O(1). \]  

**Proof.** Retaining the notation (2), we represent \(\Phi(z)\) in the form

\[ \Phi(z) = \int_0^r \frac{y(z)}{x(z - \varepsilon)}\, dx - \int_0^r \frac{x(z)}{x(z - \varepsilon)}\, dx + \int_0^\infty \frac{x(z)}{x(z - \varepsilon)}\, dx = \Phi_1(z) + \Phi_2(z) + \Phi_3(z), \]

where

\[ z = re^{i\alpha}, \quad 0 < \alpha < \infty, \quad 0 < \alpha \leq \theta \leq 2\pi - \varepsilon. \]

Furthermore,

\[ |\Phi_1(z)| \leq \frac{1}{2} \int_0^r \frac{dx}{x(z - \varepsilon)} - \frac{1}{2} \int_0^\infty \frac{dt}{t(z - \varepsilon)} = \frac{1}{2} \ln \left( 1 + \csc \frac{\alpha}{2} \right), \]

\[ |\Phi_2(z)| \leq \frac{r}{2} \int_0^\infty \frac{dx}{x(z - \varepsilon)} = \frac{1}{2} \ln \left( 1 + \csc \frac{\alpha}{2} \right). \]  

Since (1) holds for \(\Phi_2(z)\), the lemma is proved.

**Corollary.** According to the assumptions of Lemma 1,

\[ \int_0^\infty \frac{y(z)}{x(z - \varepsilon)}\, dx = -\frac{1}{2} \ln \left( 1 - \frac{z}{a} \right) + \Phi(z), \]

where \(\Phi(z)\) is a function possessing the estimate (6), while the selection of the branch \(\log z\) is subordinate to the condition \(\log 1 = 0\).

§ 3. Let us introduce a narrower class of \(A\) functions that are frequently used in applications.

**Definition.** Function \(y(x) \in A\) on \([a, +\infty)\), if:

1) \(y(x)\) is continuously differentiable on \([a, \infty)\), and \(y(a) = 1, a > 0\);

2) \(xy'(x)\) increases monotone, \(xy'(x) \geq k > 0\) on \([a, \infty)\), and \(xy'(x) \to +\infty\) for \(x \to +\infty\).*

From this definition, it follows that \(y(x)\) increases monotone on \([a, \infty)\), in which case \(y(x) \to +\infty\) for \(x \to +\infty\). Obviously, there exists a finite bound \(\omega(\infty)\) for functions of class \(A\), and the relations (1) and (6) for these functions take the form

\[ |\omega(r)| \leq [8ay'(a)^{-1}]^{-1} \quad (a \leq r \leq \infty), \]

\[ |\Phi(z)| \leq [8ay'(a)^{-1}]^{-1} + \ln \left( 1 + \csc \frac{\theta}{2} \right) \quad (a < |z| \leq \infty, 0 < \theta < 2\pi). \]

*To class \(A\) belong (smooth) logarithmic-convex functions.*