Over the principal ideal ring $k$, the Lie $k$-algebras which are free $k$-modules of finite rank are, to within isomorphism, the Lie subalgebras of the full matrix algebra $M(n, k)$.

The purpose of this paper is the extension to Lie algebras over principal ideal rings of the classical theorem of Ado on the representability of an arbitrary finite-dimensional Lie algebra over a field of characteristic zero by matrices over the same field. A Lie $k$-algebra which is representable as a submodule of the full matrix algebra $M(n, k)$ over the principal ideal ring $k$ must be a free $k$-module of finite rank. It turns out that this condition is not only necessary but sufficient for the representability of a Lie $k$-algebra, if the characteristic of $k$ is zero. The analog of Levi's theorem on the splitting off of the radical, which played an essential role in Harish-Chandra's proof of the classical theorem of Ado, is false for Lie algebras over rings. Nevertheless, the method of representation used in this paper practically coincides with the method of Harish-Chandra.

We fix the following notation: $k$ is a principal ideal ring of characteristic zero, i.e., a commutative ring with identity with no zero divisors, which contains as a subring the ring of integers, and in which every ideal is generated by one element; $F$ is the quotient field of $k$; $L$ is a Lie $k$-algebra which is a free $k$-module of finite rank; $R$ is the radical of $L$; $N$ and $N(H)$ are the nil radicals of $L$ and the algebra $H$; $L^F$ is the completion of the algebra $L$ over the field $F$; $\ast UH$ is the universal enveloping algebra of the Lie $k$-algebra $H$; $M(n, k)$ is the full matrix algebra over $k$ of rank $n^2$; $NT(n, k)$ is the subalgebra of nil triangular matrices in $M(n, k)$; $\mathcal{V}$ is the isolator of the submodule $P$ in the module $M$, i.e., the complete inverse image in $M$ of the torsion submodule in $M/P$; $\oplus$, $\bigoplus$ are, respectively, the sum in the module and in the algebra.

In proofs, induction on the rank of the module $L$ is frequently used; thus, we recall that

- a) the rank of $L$ coincides with the dimension of the vector space $F \otimes_k L$ over the field $F$;
- b) the rank of a nontrivial direct summand of a free $k$-module of finite rank is less than the rank of the extended module.

The following proposition deals with the close connection between an algebra and its completion.

**Proposition 1.** (i) The derived (lower central) series of $L^F$ is obtained by completion of the terms of the derived (lower central) series of $L$. In particular, $L$ is solvable (nilpotent) if and only if $L^F$ is solvable (nilpotent). (ii) The nil radical and radical of $L$ are isolated, and their completions coincide respectively with the nil radical and radical of $L^F$. (iii) A solvable algebra $L$ has an ideal $H$ such that $L/H \cong k$; in particular, rank $H$ is less than rank $L$.

**Proof.** (i) It is sufficient to show that $[A, B]^F = [A^F, B^F]$ for $A, B \subseteq L$. Since commutation in $L^F$ is bilinear with respect to $F$, the equation is obvious. (ii) The completion of a nilpotent ideal in $L$ is a $\ast L^F = F \otimes_k L$ is the tensor product of the $k$-modules $F$ and $L$ with the natural Lie $F$-algebra structure induced by $L$.
nilpotent ideal in the completion; therefore $N^F \subseteq N(L^F)$. On the other hand, $N(L^F) = (N(L^F) \cap L)^F \subseteq N^F$. In addition, $N = N^F \cap L$, and hence $N$ is isolated. The assertion for the radical is proved analogously.

(iii) $\sqrt{[L, L]} \neq L$; otherwise, $[L^F, L^F] = L^F$, and $L^F$ is not solvable. The ideal sought for will be, for example, the complete inverse image of a submodule with a cyclic complement in the quotient module $L/\sqrt{[L, L]}$.

Before formulating the theorem permitting a splittable extension of a representable algebra also to be represented by matrices with the aid of the given representation, we introduce the following.

**Definition.** The subalgebra $P$ of the $k$-algebra $L$ lies almost entirely in the submodule $M$, $M \subseteq L$, if there can be found a nonzero element $\alpha \in k$ satisfying the condition $\alpha P \subseteq M$.

**Theorem 1.** Let $L = H \oplus P$, where $H$ is an ideal and $P$ is a subalgebra of the Lie $k$-algebra $L$. Suppose that

1. there exist faithful representations $\rho: H \to M(r, k)$ and $\sigma: P \to M(s, k)$, where $N(H)^P \subseteq NT(r, k)$;
2. either $[H, P] \subseteq N(H)$, or $P$ lies almost entirely in $R + H$.

Then the algebra $L$ has a faithful representation $\tau: L \to M(t, k)$ such that $N(H)^T \subseteq NT(t, k)$.

**Proof.** We construct first a representation $\lambda$ of the algebra $L$ by endomorphisms of the infinite dimensional module $UH$. We set

$$ (h + p)^\lambda = R_h + D_p, $$

where $R_h: u \to uh$ is a right translation in the algebra $UH$, and $D_p$ is the derivation in $UH$ induced by the derivation $ad_H(p)$ in the Lie algebra $H$ (cf. [1], p. 171). $\lambda$ is a representation of the Lie algebra, since its restrictions to $H$ and $P$ are representations, and for $h \in H$, $p \in P$, and $u \in UH$,

$$ u(h^p p^h - p^h h^p) = u(R_h D_p - D_p R_h) = uR_{[h, p]} = u([h, p]^\lambda). $$

By the fundamental property of the universal enveloping algebra $UH$, there exists a ring homomorphism $\varphi: UH \to M(r, k)$ such that the diagram

$$ \begin{array}{ccc} H & \to & UH \\ \downarrow \sigma \downarrow \varphi \downarrow & & \downarrow \lambda \\ M(r, k) & \to & M(t, k) \end{array} $$

is commutative. Let $K$ be the kernel of $\varphi$. Let us suppose that we have found an ideal $I$ in $UH$ with the properties: 1) $I \subseteq K$; 2) $UH/I$ is a free $k$-module of finite rank; 3) $I$ is invariant with respect to the action of the endomorphisms $D_p$, $p \in P$; 4) $N(H)^m \subseteq I$ for a suitable natural $m$. Then $\lambda$ induces a representation $\bar{\lambda}$ on the quotient module $UH/I$, and $\tau$, given on $L = H \oplus P$ by the formula

$$ \tau(h + p) = \left( \begin{array}{c} \bar{\lambda}(h + p) \vdots 0 \\ \vdots \vdots \vdots \\ 0 \vdots \sigma(p) \end{array} \right), \quad h \in H, \quad p \in P, $$

will be as desired. That $\tau$ is an isomorphism is easily verified; it is only necessary to show that there exists a basis of the module $UH/I$ with respect to which $N(H)$ is represented under $\tau$ by nil triangular matrices. Using induction on the rank, it is sufficient to find a nonzero element in $UH/I$ which generates an isolated submodule and is annihilated by every element in $N(H)$. By condition 4), there exists $l$ such that $N(H)^l \subseteq I$, $N(H)^l \not\subseteq I$ (for $l = 0$ we set $N(H)^0 = UH$). Every element of $N(H)^l \setminus I$, modulo $I$, is different from zero, is annihilated by $N(H)$ under $\lambda$, and in the quotient $UH/I$, is contained in a cyclic isolated submodule which is also annihilated by $N(H)$.

Thus, it remains for us to find an ideal $I$ with the properties 1)-4). Let $G$ be the two-sided ideal in $UH$ generated by all the elements of $N(H)$ under the natural imbedding $1: H \to UH$. Since $N(1) \subseteq H$, the equation $n h = h \cdot n + [n, h]$, $n \in N(H)$, $h \in H$, shows that $G$ is generated as a left ideal by all products $n_1 n_2 \cdots n_r$, $n_i \in N(H)$, $i = 1, 2, \ldots, r$. Consequently, $G^* \subseteq K$. Let $I_1 = (G + K)^* \subseteq K$, and $I/I_1$ be the torsion submodule in $UH/I_1$. 1) $I_1 \subseteq K$, since $I_1 \subseteq K$ and $UH/K$ is without $k$-torsion. 2) $UH/I_1$ is a finitely generated