It is established that for the greatest prime factor \( P(x) \) of the value of an integral irreducible polynomial \( f(x) \) of degree \( n \geq 2 \) for integral \( x > 0 \) the estimate

\[
P(x) > c_f \ln \ln x, \quad x > x_0(f)
\]

holds, where \( c_f \) is a positive value effectively defined by the coefficients of the polynomial.

Let \( f(x) \) be a polynomial of degree \( n = 2 \) with integral rational coefficients, having distinct roots, and let \( P(x) \) be the greatest prime factor of the value of \( f(x) = 0 \) for integral rational \( x \).

Chebyshev ([1], pp. 105-106) established that the greatest prime factor of the series of numbers \( 1 + 2^2, 1 + 4^2, \ldots, 1 + (2N)^2 \) increases like a quantity of the order of \( N \). Now let \( f(x) \) be a polynomial of degree \( n = 2 \) with integral rational coefficients, having distinct roots, and let \( P(x) \) be the greatest prime divisor of the value of \( f(x) = 0 \) for integral rational \( x \). In [2] Pólya established that \( P(x) \to \infty \) as \( x \to \infty \).

Mahler [3] proved that the greatest prime divisor \( P(x) \) of the value of an integral irreducible binary polynomial \( f(x, y) \) of degree \( n \geq 3 \) at the point \( (x, y) \) increases with increase of \( X = \max (|x|, |y|) \), where \( x, y \) are mutually prime integers. A special case of this result is the fact that the greatest prime factor \( P(x) \) of the value of the integral irreducible polynomial \( f(x) \) of degree \( n \geq 3 \) at the point \( x \) increases with increase of the variable \( x \), taking integral positive values. But for a long time it was impossible to establish this rate of increase since the arguments used were ineffective. In the papers of Mahler [4] and Nagell [5] for special polynomials of degree 2 and Schinzel [6] for arbitrary polynomials of degree 2 with distinct roots the effective estimate

\[
P(x) > c_f \ln \ln x, \quad x > x_0(f),
\]

was obtained, where \( c_f > 0 \) is an effectively defined quantity, dependent on the coefficients of the polynomial. Keates [7], using a result of Baker [8] on the estimates of integral points on special elliptic curves proved that (1) also holds for third-degree polynomials. Recently Coates [10] and Sprindzhuk [11] independently of each other produced an effective proof of Mahler’s theorem [3] and obtained effective estimates for \( P(x) \)

\[
P(x) > c_f' (\ln \ln x)^{\alpha_f}, \quad x > x_1(f), \quad n \geq 3
\]

and

\[
P(x) > c_f'' \frac{\ln \ln x}{\ln \ln \ln x}, \quad x > x_2(f), \quad n \geq 5,
\]

respectively, where \( c_f' \) and \( c_f'' \) are positive quantities effectively defined by the coefficients of \( f(x) \). If we make a minor change in the final stage of Sprindzhuk’s discussion [11] (namely, putting \( s < c_P/\ln P \), which follows from the Chebyshev inequality), then the triple logarithm in (2) vanishes, and the estimate

\[
P(x) > c_f''' \ln \ln x, \quad x > x_3(f), \quad n \geq 5;
\]
will hold where $c_f > 0$ is also effectively defined. For polynomials of degree $n = 4$, relationship (3) will hold if the roots of $f(x)$ are not "exceptional" numbers in the sense of [12].

In this paper, we shall give a new definition of "exceptional" numbers, directly connected with the polynomial $f(x)$. We shall show that estimate 1 holds for polynomials "nonexceptional" by the new definition and shall establish that the class of "exceptional" polynomials in the sense of [12] does not intersect the class of "exceptional" polynomials in the new sense. This result and estimate (3) allow us to characterize the rate of increase of the greatest prime factor $P(x)$ of the value of any integral irreducible polynomial of degree $n \geq 4$ by relationship (1).

We shall give the following definitions:

The algebraic number $\theta$ of degree $n \geq 3$ is an "exceptional" number if there exists an enumeration of its conjugates $\theta(0), \ldots, \theta(n)$, such that the relationship

$$\frac{\theta(1) - \theta(i)}{\theta(0) - \theta(i)} = \frac{1 - \zeta_i}{1 - \zeta_i}$$

is satisfied for any indices $i, j$ $(i \neq j, 2 \leq i, j \leq n)$, where $\zeta_i, \zeta_j \neq 1$ are some roots of unity.

A polynomial having an "exceptional" number as a root is called an "exceptional" polynomial.

Using certain of Sprindzhuk's results [11-14], we shall prove the following:

**THEOREM.** If the integral irreducible polynomial $f(x)$ of degree $n \geq 3$ is not "exceptional" in the given sense, then estimate (1) holds for the greatest prime factor $P(x)$ of the value of $f(x)$.

The proof of the theorem mainly follows the "ineffective" scheme of Babaev and Fel'dman [9]. We shall take all our notation from [13].

Let $f(x) = p_1^{z_1} \cdots p_s^{z_s}$. We can write $f(x) = a_0 \text{N}(x - \theta)$, $a_0$ is the leading coefficient of the polynomial, $K = \mathbb{Q}(\theta)$. We put $\theta_i = a_0 \theta$ and resolve $(x - \theta_i)$ into ideals in $I_K$

$$(x - \theta_i) = a_i^{n_i} \cdots p_i^{n_i},$$

and the prime ideals $p_1, \ldots, p_s$ enter into the prime numbers $p_1, \ldots, p_s$. We may assume that the ideals $p_i$ enter into the corresponding prime numbers $p_i$ [13]. Then

$$a | a_0^{n_i - 1},$$

such that $a_i^{n_i} \cdots p_i^{n_i} = a | a_0^{n_i} \mathbb{D}(\theta_i)^{n_i}.$

If $h$ is the number of classes of ideals of the field $K$, then $p_i^h = (p_i)$ is a principal ideal of the field and $p_i \in \mathbb{Z}_K$. Let $U_i = h u_i + r_i, 0 \leq r_i < h, i = 1, \ldots, s$. The ideal $a_i^{n_i} \cdots p_i^{n_i}$ is the ratio of principal ideals; hence, it is a principal ideal $(\rho)$, consequently,

$$(x - \theta_i) = (\rho_1^{v_1} \cdots \rho_s^{v_s}).$$

Hence, we have

$$x - \theta_i = \rho_1^{v_1} \cdots \rho_s^{v_s} u_1^{v_1} \cdots u_k^{v_k} = \mu,$$

where $e_1, \ldots, e_k$ are basic units of the field $K$, $u_i \geq 0, \ldots, u_k \geq 0$, $v_1, \ldots, v_k$ are rational integers. Taking conjugates in (5) and eliminating $x$, we obtain the system of equations

$$\frac{\theta(0) - \theta(i)}{\mu} = 1 - \frac{\mu^{(i)}}{\mu^{(0)}}$$

$$(i = 2, \ldots, n).$$

Eq. (6) allows us to write down that

$$\left| 1 - \frac{\mu^{(i)}}{\mu^{(0)}} \right| = \left| 1 - \frac{\mu^{(i)}}{\mu^{(0)}} \right| = \left| \left( \frac{\mu^{(0)}}{\mu^{(i)}} \right)^{v_1} \cdots \left( \frac{\mu^{(0)}}{\mu^{(i)}} \right)^{v_k} \right|.$$