The compactness property of a family of functions harmonic for a Markov process is studied and, in particular, an inequality of Harnack type is derived. It is shown that under broad conditions the property that a function be locally harmonic implies that it is harmonic.

The present paper is a direct continuation of [1] in which explanations regarding our notation and terminology may be found. According to the basic result of [1], under broad conditions any uniformly bounded sequence of functions which are harmonic with respect to a Markov process contains a subsequence which converges everywhere in the phase space of the process to some function harmonic with respect to the process. Is it possible to extract from the original sequence a subsequence which converges to the limit function uniformly on any compact set of phase space? An affirmative answer can be given to this question under specific hypotheses (see below, Theorem 1) in complete correspondence to the classical theorem of Harnack and in analogy to the theorem of Mokobodskii (cf. [3], p. 22) pertaining to modern potential theory. Using this result, for nonnegative functions which are harmonic with respect to a broad class of continuous Markov processes, we obtain an analogue of the classical inequality of Harnack (Theorem 2).* These assertions are derived from the result of [1] just mentioned by using certain modifications of the methods of modern potential theory.

To some extent Theorem 3 stands by itself; it gives conditions under which the property that a function be locally harmonic implies that it is harmonic for a Markov process.

1. In this section we consider the compactness properties of a family of functions which are harmonic for a Markov process. We fix to the end of the work a locally compact Hausdorff space $E$ with a countable basis and a standard Markov process $X = (x_t, \mathcal{L}_t, \mathbb{P}_x)$ in $E$. A function $f: V \to \mathbb{R}$ continuous in $V$ is called harmonic for $X$ in the open set $V \subset E$ if it is continuous in $V$ in the intrinsic topology (cf. [5]) generated by the process $X$ and if for any compact set $K \subset V$

$$ f(x) = \mathcal{M}_f(x_t(K)) \quad (x \in V), $$

where $T(K)$ is the moment of the first exist of the trajectory of the process from $K$ after $+0$. A function harmonic for the process $X$ is defined as a function harmonic for $X$ in the set $V = E$.

We introduce the following condition:

(A) Any bounded function $f \geq 0$ which is harmonic for $X$ in an arbitrary set $V \subset E$ is continuous in $V$.

THEOREM 1. Suppose that the standard process $X$ satisfies condition (A) and Meyer's condition (L) (cf. [1]). Then from any sequence of functions harmonic for $X$, $\{f_n; n \geq 1\}$ which is uniformly bounded in $E$ it is possible to extract a subsequence which converges for $n \to \infty$ to some function harmonic for $X$ uniformly on each compact set $K \subset E$.

The proof follows the same outline as that of the theorem of Mokobodskii mentioned above.


†If the process $X$ is continuous, then it is sufficient to require that the function $f$ be defined only in $V$.

We shall first assume that the $f_n$ are nonnegative. According to Theorem 1 of [1], from the sequence \{\{f_n\}\} we can extract a subsequence \{\{f_n^\#\}\} which converges everywhere in $E$ to some function $f$ which is harmonic for $X$. In order not to complicate the notation, we shall assume that $f_n^\# = f_n(n \geq 1)$, which does not affect the generality of the argument. We need the identity

$$f = \sup_{n \geq 1} \tilde{h}_n,$$

(2)

where $h_n = \inf_{m \geq n} f_m$, and for any function $h$ on $E$ the symbol $\tilde{h}(x)$ means $\lim_{y \to x} h(y)$. To prove (2), we note that for $x \in E$

$$f(x) = \lim_{n \to \infty} f_n(x) = \sup_{n \geq 1} \inf_{m \geq n} f_m(x) = \sup_{n \geq 1} h_n(x) \geq \sup_{n \geq 1} \tilde{h}_n(x),$$

(3)

[The presence of the inequality here follows from the upper semicontinuity of the function $h_n$, which is the limit of a nonincreasing sequence of continuous functions $g_k = \inf \{f_n, \ldots, f_n+k\}$ ($k \geq 1$).] On the other hand, if the open set $U \subset E$ has compact closure and $x \in U$, then in correspondence with the definition of a function harmonic for the process

$$f(x) = \mathcal{M}_x f(x) = \mathcal{M}_x \{\lim h_n(x)\} = \sup_{n \geq 1} \mathcal{M}_x h_n(x),$$

(4)

where the theorem on the monotonic passage to the limit under the integral sign has been used, and we have set $\tau = T(U)$. The function $\mathcal{M}_x h_n(x)$ is harmonic in $U$ and does not exceed $\mathcal{M}_x f_m(x) = f_m(x)$ for $m > n$. From this and the fact that it is continuous in $U$ [see condition (A)] we see that it also does not exceed the function $\inf_{m \geq n} f_m(x) = h_n(x)$ in $U$. Therefore, on the basis of (4), we may write

$$f(x) = \sup_{n \geq 1} \mathcal{M}_x h_n(x) \leq \sup_{n \geq 1} \tilde{h}_n(x) \quad (x \in U).$$

(5)

To obtain the identity (2) it remains to compare (3) and (5).

By their definition the functions $\tilde{h}_n$ are lower semicontinuous, do not decrease with increasing $n$, and their limit, according to (2), is a continuous function $f$. Therefore, for $n \to \infty$ they tend to $f$ uniformly on each compact set $K \subset E$, as follows from the appropriate version of the well known theorem of Dini. But then, obviously, the function $h_n$ have the same property. Thus, if $\varepsilon > 0$ is fixed, then for all sufficiently large values of $n$

$$f_n(x) \geq h_n(x) > f(x) - \varepsilon,$$

(6)

on the fixed compact set $K \subset E$.

In this argument let us replace the function $f_n(x)$ by the functions

$$H_n(x) = C \cdot \mathcal{P}_x \{\tau < t\} - f_n(x) \geq 0,$$

where $\tau = T(U)$, $U$ is some relatively compact neighborhood of the compact set $K$, and $C = \sup_{n \geq 1; x \in K} f_n(x)$. As a result, in place of (6), we obtain the relation

$$C \cdot \mathcal{P}_x \{\tau < t\} - f_n(x) \geq C \cdot \mathcal{P}_x \{\tau > t\} - f(x) - \varepsilon,$$

i.e., $f_n(x) < f(x) - \varepsilon$ for all $x \in K$, if $n$ is sufficiently large. The last inequality together with (6) establishes the fact that $f_n$ tends uniformly to $f$ for $n \to \infty$.

Thus, we have dealt with the case of nonnegative functions $f_n$. The general case reduces to this, since the functions $f_n$ can always be replaced by harmonic functions $f_n^\# = f_n - g \geq 0$, where $g = C_1 \cdot \mathcal{P}_x \{\Omega_1\}$, $C_1 = \inf_{n \geq 1; x \in K} f_n(x)$, and $\Omega_1$ is the event that the trajectory $x_t$ passes out of any compact $K$ in $E$ at least once.

**Remark 1.** Condition (A) and Meyer's condition (1) are satisfied in the case of a continuous, homogeneous strong Feller process which is uniformly stochastically continuous on each compact set of the space $E$ (cf. [5], Theorem 13.2). In particular, these conditions are satisfied for a broad class of diffusion processes.