CONCEPTS OF THE SELF-ADJOINTNESS OF A QUASI-ELLiptic OPERATOR

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We prove the following theorem for the operator \( L = \sum_{k=1}^{n} (-1)^{m_k} D_k^{m_k} + q \) considered in \( L_2(\mathbb{R}^n) \) (the \( m_k \) are natural numbers): If \( q(x) > -C \max_k |x_k|^{1-1/2m_k} (C > 0) \) for sufficiently large \( |x| \), then \( L \) is a self-adjoint operator.

In the complex Hilbert space \( L_2(\mathbb{R}^n) \) we consider a quasi-elliptic operator of the form

\[
L = \sum_{k=1}^{n} (-1)^{m_k} D_k^{m_k} + q(x),
\]

where \( D_k^{m_k} = \frac{\partial^{2m_k}}{\partial x_k^{2m_k}} \), the \( m_k \) are natural numbers, and \( q(x) \) is a continuous function defined on \( \mathbb{R}^n \).

**Lemma 1.** For every \( q(x) \in C(\mathbb{R}^n) \) there is an infinitely differentiable function \( \tilde{q}(x) \in C^\infty(\mathbb{R}^n) \) such that \( |q(x) - \tilde{q}(x)| < 1 \) for all \( x \in \mathbb{R}^n \).

**Proof.** Suppose that the set of functions \( \{u_j(x)\} \) is a partition of unity, that is, each \( u_j(x) \) is concentrated in a cube \( K_j \) and belongs to the class \( C^\infty(\mathbb{R}^n) \), \( \sum_{j=1}^{\infty} u_j(x) = 1 \), and every point \( x \in \mathbb{R}^n \) belongs to at most \( N \) of the sets \( K_j \). Then \( q(x) = \sum_{j=1}^{\infty} q_j(x)u_j(x) = \sum_{j=1}^{\infty} q_j(x) \), where \( q_j(x) = q(x)u_j(x) \). For every \( q_j(x) \) we can find a \( \tilde{q}_j(x) \in C^\infty(\mathbb{R}^n) \) such that

1) \( |q_j(x) - \tilde{q}_j(x)| < 1/N \) for all \( x \in \mathbb{R}^n \),
2) \( \tilde{q}_j(x) \) is concentrated in \( K_j \).

We put

\[
\tilde{q}(x) = \sum_{j=1}^{\infty} \tilde{q}_j(x)
\]

(the series is convergent because for every \( x \) the sum contains not more than \( N \) nonzero summands). Then for every \( x \in \mathbb{R}^n \) we have

\[
|q(x) - \tilde{q}(x)| = \left| \sum_{j=1}^{\infty} (q_j(x) - \tilde{q}_j(x)) \right| < \sum_{j=1}^{\infty} |q_j(x) - \tilde{q}_j(x)|.
\]

The last sum contains not more than \( N \) nonzero summands, and they are all less than \( 1/N \).

Therefore, \( |q(x) - \tilde{q}(x)| < N \cdot 1/N = 1 \).

Since a defect number is stable under bounded perturbations, we may further assume that \( q(x) \in C^\infty(\mathbb{R}^n) \).

Let \( L \) be the minimal operator generated by (1) in \( L_2(\mathbb{R}^n) \), and let \( L^* \) denote the adjoint of \( L \). We denote the domains of \( L \) and \( L^* \) by \( D_L \) and \( D_{L^*} \), respectively. Let \( L' \) be the re-
striction of \( L^* \) such that
\[
D_{L^*} = \{ u; \ u \in C^\infty(R^n) \cap L^2(R^n), \ l(u) \in L^2(R^n) \}.
\]
The following lemma holds.

**Lemma 2.** The adjoint operator \( L^* \) is the closure of its restriction to the set of infinitely differentiable functions \( u \in D_{L^*} \).

**Proof.** As is well known (see [1])
\[
D_{L^*} = D_L + N_i + N_{-i},
\]
where \( N_i \) and \( N_{-i} \) are defect subspaces of \( L \) that consist of the solutions of the equations \( L(u) = iu, \ L(u) = -iu \), respectively, that belong to \( L^2(R^n) \).

It is well known that if \( q(x) \) is infinitely differentiable, then the solutions of these equations are also infinitely differentiable (see [2]). We denote by \( L_0 \) an operator with the domain \( C^\infty(R^n) \), then the minimal operator \( L \) is the closure of \( L_0 \).

Let \( u \in D_{L^*} \); then by (2) we have \( u = u_0 + u(i) + u(-i) \), where \( u_0 \in D_{L^*}, \ u(i) \in N_i, \ u(-i) \in N_{-i} \), and \( L^*u = L_0u - iu(i) + iu(-i) \).

Since a minimal operator is the closure of an operator defined on a set of infinitely differentiable finite functions, there is a sequence \( \{u_n\} \) of infinitely differentiable finite functions \( u_n \) such that \( u_n \to u_0, \ L_0u_n \to L_0u \) as \( n \to \infty \). Then \( v_n = u_n + u(i) + u(-i) \to u, \ v_n \in D_{L^*} \) and
\[
L^*v_n = L^*(u + u(i) + u(-i)) = L_0u - iu(i) + iu(-i),
\]
\[
\lim_{n \to \infty} L^*v_n = L_0u - iu(i) + iu(-i) = L^*v.
\]
Thus, \( L^* \) is the closure of \( L' \).

It follows from Lemma 2 that to prove that \( L \) is self-adjoint it is sufficient to show that \( L' \) is symmetric on its domain.

By a layer \( P \) we mean the difference of two identically arranged rectangular parallelepipeds with edges parallel to the coordinate axes and with the center of symmetry at the origin. We denote by \( h_k \) the thickness of \( P \) in the direction of the \( x_k \) axis.

**Lemma 3.** Let \( \{P_\nu\} \) be an unboundedly extending sequence of disjoint layers of thickness \( h_\nu, k \) in the direction of the \( x_k \) axis. If the following relation holds for every \( u \in D_{L^*} \).
\[
\lim_{\nu \to \infty} \int_{P_\nu} h_{\nu, k}^{-2m_\nu} |D^4 u|^2 \, dx = 0 \quad (j \leq m_k, k = 1, \ldots, n),
\]
then \( L \) is self-adjoint.

**Proof.** We denote by \( K_\nu \) and \( K_{\nu+1} \), those parallelepipeds for which \( P_\nu = K_{\nu+1} \setminus K_\nu \). For each \( \nu \) we construct an infinitely differentiable function \( \psi_\nu(x) \) such that \( \psi_\nu(x) = 1 \) if \( x \in K_\nu, \psi_\nu(x) = 0 \) if \( x \notin K_{\nu+1}, 0 \leq \psi_\nu(x) \leq 1, \) and \( |D^k \psi_\nu| \leq C h_{\nu, k}^{-3} \), where \( C > 0 \). For \( \psi_\nu(x) \) we can take \( \prod_{k=1}^n \psi_{\nu, k}(x_k) \), where the \( \psi_{\nu, k}(x_k) \) are chosen in the same way as in the one-dimensional case (see [3]). We put \( \psi_\nu(x) = 1 - \psi_\nu(x) \).

We prove that the operator \( L' \) is symmetric. For any two functions \( u, v \in D_{L^*} \) we have
\[
(L'u, v) = (u, L'v) = (l(u\psi_\nu), v) - (u, l(v)) = (l(u\psi_\nu), v) + (l(u\psi_\nu), v) - (u, l(v)).
\]
The function \( u\psi_\nu \) is finite, therefore \( l(u\psi_\nu), v \to (u, l(v)) \to 0 \) as \( \nu \to \infty \). We only need to prove that
\[
\lim_{\nu \to \infty} (l(u\psi_\nu), v) = 0
\]
\[
(l(u\psi_\nu), v) = (\psi_\nu l(u), v) + \int_{P_\nu} \sum_{|\alpha|=4} \sum_{|\beta|=3} \sum_{j=1}^{2m_\nu} \sum_{i=1}^{2m_\nu} C_{\alpha, \beta, \sigma}^i D_{\nu, i}^{2m_\nu-\alpha-j} u D_{\nu, i}^{2m_\nu-\beta} v \, dx.
\]
Clearly, \( (\psi_\nu l(u), v) \to 0 \) as \( \nu \to \infty \). We transform the second summand by integrating by parts, and estimate it by using the inequality \( 2|ab| \leq a^2 + b^2 \); as a result we obtain

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