INVERSION OF THE OSCILLATORY PROPERTY OF FOCUSING OPERATORS

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Suppose that $E$ is a real Banach space with a cone $K$ and suppose that the homogeneous additive operator $A$ that is positive on $K$ is focusing, i.e., $AK \subset K_{\infty, \mu}$ for certain $\mu \in K$ and $\rho > 1$. Then, as is well known, the operator $A$ uniformly reduces the oscillation (osc) between the elements of $K$. In this paper we show that only the focusing operators have this property.

In this paper we study homogeneous additive operators that are positive on the cone of a Banach space. We prove that the basic result of [1] can be inverted. We give an equivalent description of the class of focusing operators.

We use the basic terminology of [2, 3].

1. Suppose that $E$ is a real Banach space with a cone $K$. We recall that (according to [1]) the quantity

$$\theta(x, y) = \inf \{\beta/\alpha : \alpha x \leq y \leq \beta x\}$$

is called the deviation $\theta(x, y)$ between the elements $x, y \in K$. If $\theta(x, y) < \infty$, then the elements $x, y$ are said to be commensurable (belong to the same component of the cone $K$). The basic properties of the deviation were given in [1, 4]. We note that the collection of rays of any component is metrizable by the formula $r(\xi, \eta) = \ln \theta(x, y)$, where $x$ and $y$ are any nonzero elements of the rays $\xi$ and $\eta$, respectively.

Suppose that $A$ is a homogeneous additive operator that is positive on the cone $K$. The quantity

$$\theta(A) = \sup_{x,y \in K} \sqrt{\theta(Ax, Ay)}$$

is called the $\theta$ norm of the operator $A$. Here, naturally, the least upper bound is taken only over those elements $x, y \in K$ that have nonzero images $Ax, Ay$.

The operator $A$ is, by definition, focusing if $\theta(A) < \infty$. For example, any integral operator

$$Ax(t) = \int_{0}^{t} K(t, s)x(s)ds$$

that acts in some functional space $E_{0}$ is focusing if its kernel satisfies the inequalities $0 < m \leq K(t, s) \leq M < \infty$ for all $t, s \in \Omega$ (see [1, 5]). More generally, operator (3) is focusing if and only if, for some $\rho$,

$$K(t, s) \leq \rho K(t, \sigma)K(\tau, s).$$

The square of the integral operator that inverts the linear interpolational boundary-value problem on an interval of nonoscillation (see [6, 1]) is an important example of a focusing operator.

Suppose that $x \in K$ ($x \neq 0$) and $y \in E_{x}$, i.e., $-tx \leq y \leq tx$ for some finite $t = t(y)$. The quantity

$$\osc(y : x) = \inf \{\beta - \alpha : \alpha x \leq y \leq \beta x\}$$


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is called the oscillation of the element $y$ with respect to $x$. If, e.g., $x(t), y(t) \in E_\Omega$ and $x(t) \geq 0$, then

$$\text{osc}(y : x) = \inf_{t,s} \left[ \frac{y(t)}{x(t)} - \frac{y(s)}{x(s)} \right].$$

Here the infimum is taken over those $t,s \in \Omega$ for which $x(t), x(s) > 0$. This stipulation is unessential since the inclusion $y \subseteq E_x$ implies that $[x(t) = 0] + [y(t) = 0]$.

Any operator that is positive on the cone $K$ does not increase the oscillation between the elements. According to [1] any focusing operator $A$ uniformly decreases the oscillation, viz.,

$$\text{osc}(Ay : Ax) \leq \frac{\theta(A) - 1}{\theta(A) + 1} \text{osc}(y : x)$$

for any $x \geq 0 (x \neq 0)$ and $y \subseteq E_x$. For integral operators with positive kernels a similar property was established in [6]. In [1] it was noted that the estimate (4) is best possible in the class of focusing operators.

At first sight the class of operators that uniformly decrease the oscillation seems to be more extensive than the class of focusing operators. In what follows we prove that these classes coincide. Moreover, the property (4) is inverted exactly.

**THEOREM 1.** Suppose that the homogeneous additive operator $A$ is positive on the cone $K$. Suppose that for the constant $q < 1$ and any $x \in K$ with nonzero image $Ax$

$$\text{osc}(Ay : Ax) \leq q \text{osc}(y : x) \quad (y \subseteq E_x).$$

Then $\theta(A) < \infty$, i.e., the operator $A$ is focusing.

For the focusing operator $A$ we denote by $\text{osc} A$ (as in [4]) the least of the numbers $q$ that satisfy (5).

**THEOREM 2.** Under the conditions of Theorem 1

$$\theta(A) = \frac{\text{osc} A + 1}{\text{osc} A - 1}.$$  

Thus, estimate (4) is best possible for each focusing operator.

The proof of the above theorems requires the study of special geometrical properties. We introduce some notation. We denote by $\sup(y : x)$ the least value of $t$ that satisfies the inequality $y \leq tx$ (of course, if $x \in K$). Analogously, we denote by $\inf(y : x)$ the greatest number $s$ that satisfies the inequality $sx \leq y (x \in K)$. Then for $x, y \in K$

$$\text{osc}(y : x) = \sup(y : x) - \inf(y : x),$$

$$\theta(x, y) = \frac{\sup(y : x)}{\inf(y : x)} = [\sup(y : x)]/[\sup(x : y)].$$

The nonzero elements $x, y \in K$ are said to be $K$-disjunctive if $\inf(x : y) = \inf(y : x) = 0$. For any $K$-disjunctive $x, y \in K$ the elements $\alpha x$ and $\beta y$ are also $K$-disjunctive for any $\alpha, \beta > 0$.

**LEMMA 1.** For $K$-disjunctive elements $x, y \in K$

$$\sup(x + ty : x + y) = 1$$

for any $t \in [0, 1]$.

**Proof.** We assume the contrary. Suppose that

$$\gamma = \sup(x + ty : x + y) \neq 1$$

for some $t \in [0, 1]$. Since by the definition of $\gamma$

$$x + ty \leq \gamma (x + y),$$

it follows that $\gamma < 1$ and $t < 1$. Moreover, by virtue of (9) $(1 - \gamma) x \leq (\gamma - t) y$. This implies at once that $(\gamma - t) > 0$ and, moreover,

$$\inf(y : x) \geq \frac{1 - \gamma}{\gamma - t} > 0.$$