Thus, inequality (17) holds for any elements \( x, y \in K \) with nonzero images \( A_x \) and \( A_y \). Therefore, the operator \( A \) is focusing. By virtue of (2), (17) implies that
\[
\theta(A) \leq \frac{1+q}{1-q}.
\]

The last inequality together with (4) proves (6).

This completes the proof of Theorems 1 and 2.

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LITERATURE CITED


AXIOMATIC THEORY OF CONVEXITY

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The axiomatic construction of the theory of convexity proceeds from an arbitrary set \( M \) and a mapping \( \mathcal{L} : M^2 \to 2^M \), i.e., from a pair \((M, \mathcal{L})\). It is shown that such a space of a certain type is domain finite. A condition is given which, for such spaces, implies join-hull commutativity. A connection is established between the Carathéodory number and join-hull commutativity. Conditions are given which imply a separation property of the space \((M, \mathcal{L})\). Convexity spaces which are domain finite are characterized.

1. A convexity space of type 2 (abbreviated c.s. of type 2) is a set \( M \) together with a mapping \( \mathcal{L} : M \times M \to 2^M \) which satisfy the following axioms:
   1. \( \mathcal{L}(a, b) = \mathcal{L}(b, a) \),
   2. \( \mathcal{L}(a, a) = \{a\} \),
   3. \( \{a, b\} \subseteq \mathcal{L}(a, b) \) , if \( a \neq b \),
   4. \( c, d \subseteq \mathcal{L}(a, b) \Rightarrow \mathcal{L}(c, d) \subseteq \mathcal{L}(a, b) \).

   The elements of the set \( M \) are called points. The set \( \mathcal{L}(a, b) \) is called the segment joining the points \( a \) and \( b \).

   A nonempty subset \( Q \subseteq M \) is called \( \mathcal{L} \) convex if \( a, b \in Q \) implies \( \mathcal{L}(a, b) \subseteq Q \). The empty set is \( \mathcal{L} \) convex by definition.
Clearly, a one-element set, a segment, and M itself are L convex.

If \( \{Q_i\}_{i \in I} \) is a family of L convex sets, then it is easy to verify that \( \bigcap_{i \in I} Q_i \) is L convex. Suppose \( S \subseteq M \). By the L convex hull of \( S \), written \( \text{conv}_L(S) \), we mean the smallest L convex set containing \( S \). Clearly, \( \text{conv}_L(S) \) exists for any \( S \subseteq M \), and \( \text{conv}_L(S) = \bigcap \{ C : C \) is L convex, \( C \supseteq S \} \). It is obvious that \( \text{conv}_L(a, b) = L(a, b) \).

We define the dimension of a c.s. of type 2.

**Definition.** We say that \( (M, L) \) is finite-dimensional if there exists a natural number \( n_0 \) such that if \( T = \{a_1, \ldots, a_m\} \) is a finite subset of \( M \), then
\[
\text{conv}_L(T) = \bigcup \{ \text{conv}_L(S) : S \subseteq T, |S| \leq n_0 + 1 \}.
\]
We then say that \( (M, L) \) is at most \( n_0 \)-dimensional.

If \( (M, L) \) is finite-dimensional, then we say that its dimension is exactly equal to \( n \) if it is at most \( n \) and \( n \) is the smallest natural number with this property.

Following D. C. Kay and E. W. Womble [1], a convexity structure for \( X \) is defined to be a family \( \mathcal{G} \) of subsets of \( X \) together with the pair \( (X, \mathcal{G}) \), called a convexity space, if the following two conditions are satisfied:

a) \( \emptyset \) and \( X \) belong to \( \mathcal{G} \);

b) \( \bigcap \mathcal{F} \subseteq \mathcal{G} \) for each subfamily \( \mathcal{F} \subseteq \mathcal{G} \).

If c) \( \{x\} \subseteq \mathcal{G} \) for each \( x \in X \) then \( \mathcal{G} \) is called \( T_1 \).

By the hull operator corresponding to the convexity structure \( \mathcal{G} \) we mean the operator
\[
\mathcal{G}(S) = \bigcap \{ C : C \subseteq \mathcal{G}, C \supseteq S \}, S \subseteq X.
\]

It is easy to see that it possesses the following properties:

1) \( S \subseteq \mathcal{G}(S) \) for \( S \subseteq X \); ii) if \( S_1 \subseteq S_2 \) then \( \mathcal{G}(S_1) \subseteq \mathcal{G}(S_2) \); iii) \( \mathcal{G}(\mathcal{G}(S)) = \mathcal{G}(S) \); iv) if \( \mathcal{G}(S) = S \), then \( S \subseteq \mathcal{G} \).

The set \( \mathcal{G}(S) \) is called the \( \mathcal{G} \) hull of \( S \), and \( S \) is called \( \mathcal{G} \) convex if \( \mathcal{G}(S) = S \).

If \( \mathcal{G}(S) = \bigcup \{ \mathcal{G}(T), T \subseteq S, |T| < \infty \} \) for each \( S \subseteq X \), then the convexity structure is called domain finite.

A convexity structure \( \mathcal{G} \) has Carathéodory number \( c \) if \( c \) is the smallest natural number with the following property: the \( \mathcal{G} \) hull of any set \( S \subseteq X \) is the union of the \( \mathcal{G} \) hulls of those subsets \( S \) for which \( |T| \leq c \). Also, a convexity structure has Helly number \( h \) if \( h \) is the smallest natural number such that a finite subfamily \( \mathcal{F} \) of sets in \( \mathcal{G} \) has nonempty intersection if each \( h \) members of \( \mathcal{F} \) have nonempty intersection, and Radon number \( r \) if \( r \) is the smallest natural number such that any set \( S \), \( |S| \geq r \), has a Radon partition, that is, can be partitioned into two nonempty subsets \( (S_1, S_2) \) such that \( \mathcal{G}(S_1) \cap \mathcal{G}(S_2) \neq \emptyset \).

If \( \mathcal{G} \) is the family of convex sets in Euclidean space \( \mathbb{E}^d \) of dimension \( d \), then the classical theorems of Carathéodory, Helly, and Radon show that \( \mathcal{G} \) has \( c = h = d + 1 \) and \( r = d + 2 \).

If \( (M, L) \) is a c.s. of type 2, then it is easy to see that it is a \( T_1 \) convexity structure. Moreover, as we will now show, it is domain finite.

**THEOREM 1.** If \( (M, L) \) is a c.s. of type 2, then it is domain finite.

**Proof.** We must show that for any \( S \subseteq M \),
\[
\text{conv}_L(S) = \bigcup \{ \text{conv}_L(T) : T \subseteq S, |T| < \infty \}.
\]
We will establish two inclusions:

1. \( \bigcup \{ \text{conv}_L(T) : T \subseteq S, |T| < \infty \} \subseteq \text{conv}_L(S) \).
2. \( \text{conv}_L(S) \subseteq \bigcup \{ \text{conv}_L(T) : T \subseteq S, |T| < \infty \} \).

The first is obvious. To prove the second it suffices to show that \( C = \bigcup \{ \text{conv}_L(T) : T \subseteq S, |T| < \infty \} \) is \( L \) convex and that \( C \supseteq S \). It is easy to see that \( C \supseteq S \) is \( L \) convex. Suppose \( a, b \in C \). Then \( a \in \text{conv}_L(T_1), T_1 \subseteq S, |T_1| < \infty \) and \( b \in \text{conv}_L(T_2), T_2 \subseteq S, |T_2| < \infty \). Consider \( \text{conv}_L(T_1 \cup T_2) \). Clearly, \( T_1 \cup T_2 \subseteq S, |T_1 \cup T_2| < \infty \). It is also obvious that \( a, b \in \text{conv}_L(T_1 \cup T_2) \). Therefore, \( a, b \in C \).