ON THE DIMENSION OF GRADED ALGEBRAS

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To each graded algebra $R$ with a finite number of generators we associate the series $T(R, z) = \sum d_n z^n$, where $d_n$ is the dimension of the homogeneous component of $R$. It is proved that if the dimensions $d_n$ have polynomial growth, then the Krull dimension of $R$ cannot exceed the order of the pole of the series $T(R, z)$ for $z = 1$ by more than 1.

We shall mainly consider the graded algebra $R = \sum_{i=0}^{\infty} R_i$ over an arbitrary field $k$ having a finite number of generators and a finite number of defining relations. However, we shall require a somewhat more general formulation of the main hypothesis.

Let $M = \sum_{i=0}^{\infty} M_i$ be a graded bimodule over a free algebra $k(x_1, \ldots, x_s)$ having a finite number of generators and a finite number of defining relations. Let the symbol $d_i(M)$, or simply $d_i$, denote the dimension of $M_i$ as a vector space over $k$. The main hypothesis has three equivalent formulations.

1. There exists a linear recurrence relation with integral coefficients $\lambda_i$:

$$d_n = \sum_{i=1}^{k} \lambda_i d_{n-i},$$

which is valid for all sufficiently large $n$. The polynomial $Q(M, z) = z^k - \sum \lambda_i z^{k-i}$ will be called the characteristic polynomial of the module $M$.

2. For all sufficiently large $n$

$$d_n = \sum P_i(n) a_i^n,$$

where the $a_i$ are algebraic integers and the $P_i(n)$ are polynomials with rational coefficients.

3. The series

$$T(M, z) = \sum_{n=0}^{\infty} d_n z^n$$

is a rational function with integral coefficients and lowest term in the denominator equal to 1.

The proof that these formulations are equivalent can be found in [1], Chap. 3.

The foregoing quantities are related as follows. The $a_i$ are the roots of the characteristic polynomial $Q(M, z)$; $P_i(n)$ is a polynomial of degree $k_i$, where $k_i + 1$ is the multiplicity of $a_i$; $T(M, z) = f(z)g^{-1}(z)$, where $g(z) = z^kQ(M, z^{-1})$, and the degree of $f(z)$ is less than the number starting with which the recurrence relation holds for all following numbers.

In the general case the main hypothesis remains unproved. A proof exists in the following special cases: modules over commutative algebras ([2], Theorem 15.2); algebras defined by a finite collection of words ([3], Theorem 2); algebras having global dimension less than 3 ([3], Theorem 3). Theorem 1 below proves the main hypothesis for a rather narrow class of algebras; however, it is interesting because it establishes a connection between the coefficients of recurrence relation (1) and the Möbius function for a monoid (see [4]) or a partially ordered set (see [1], 2.2).
Following Cartier and Foata [4], we introduce the following definitions. Let $M$ be a monoid and $M^*$ the set of nonidentity elements of $M$. By a factorization of $x$ in $M$ we mean any sequence $(x_1, x_2, \ldots, x_n)$ of elements of $M^*$ such that $x = x_1x_2\ldots x_n$. The number $n$ is called the length of the factorization. We consider a monoid in which any element admits only a finite number of distinct factorizations. The number of factorizations of $x$ is denoted by $d(x)$. The number of factorizations of even length is denoted by $d_+(x)$ and of odd length by $d_-(x)$. It is obvious that $d(x) = d_+(x) + d_-(x)$. The function $\mu(x) = d_+(x) - d_-(x)$ is called the Möbius function of $M$.

Let $A$ be the set of functions on the monoid $M$. We introduce a multiplication operation on $A$ as follows:

$$fg(x) = \sum_j f(x_j) g(x_2),$$

which turns $A$ into a monoid. The function $\varepsilon(x), \varepsilon(1) = 1$, and $\varepsilon(x) = 0$ for $x \neq 1$ is the identity of $A$. The sum in (4) can be extended to all pairs $(x_1, x_2)$ such that $x = x_1x_2$; among these pairs are $(1, x)$ and $(x, 1)$. Let $\delta(x) = 1$ for all $x \in M$. Then (see [4])

$$\zeta = \mu \delta = \varepsilon.$$

**Theorem 1.** Let $M$ be a graded monoid without zero and with a finite number of generators. Let $\mu(x)$ be such that $\mu(x) \neq 0$ for only a finite number of elements of $M$. Then the semigroup algebra $k(M)$ satisfies the main hypothesis.

**Proof.** Consider the following series product:

$$\sum_{x \in M} \mu(x) x \sum_{y \in M} \zeta(y) y = \sum_{x \in M} \varepsilon(x) x = 1. $$

Since there is no zero in $M$,

$$\sum \mu(x) t|x| \sum \zeta(y) t|y| = 1,$$

where $|x|$ is the degree of $x$. Collecting the coefficients of $t^N$, we obtain

$$\sum \mu(x) \zeta(y) = 0, \quad (5)$$

where $|x| + |y| = N > 0$.

Let $\lambda_i = \sum_{|x| = i} \mu(x)$ and let $d_j = \sum_{|y| = j} \zeta(y)$ be the number of elements of $M$ having degree $j$. These elements can be taken as a basis for the $j$-th homogeneous component of the semigroup algebra $k(M)$. Therefore $d_j$ coincides with $d_j(k(M))$.

Equality (5) can now be rewritten in the form

$$\sum_{i=0}^N \lambda_i \zeta(y) = \sum_{i=0}^{N} \mu(x) \sum_{j=0}^{N-i} \zeta(y) = \sum_{j=0}^{N} \lambda_j d_{N-i} = 0. \quad (6)$$

Since almost all $\lambda_j = 0$, by the condition of the theorem, equality (6) becomes recurrence relation (1).

**Corollary.** Let an algebra $R$ be defined by a finite number of generators $x_1, x_2, \ldots, x_s$ and relations of the form $x_1x_j - x_jx_1$ for some $i$ and $j$. Then $R$ satisfies the main hypothesis.

**Proof.** The algebra $R$ satisfies all the conditions of Theorem 1 (see [4]).

The basic ring-theoretic operations, when applied to algebras satisfying the main hypothesis, again result in algebras satisfying it.

a) For any exact sequence

$$0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0,$$

$$T(M, z) = T(M_1, z) + T(M_2, z).$$

b) It is obvious that

$$T(M_1 \oplus M_2, z) = T(M_1, z) \cdot T(M_2, z).$$

c) If $R_1$ and $R_2$ are graded algebras and $R_1 \ast R_2$ is their free product, then

$$T(R_1 \ast R_2, z) = [T^{-1}(R_1, z) + T^{-1}(R_2, z) - 1]^{-1}. \quad (7)$$