ON UPPER AND LOWER VALUES OF A GENERALIZED FUNCTION AT A POINT

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In this paper we introduce, for generalized functions of $D'(\mathbb{R}_1)$, the concepts of functions of upper and lower values, and we give a complete description of them for generalized functions belonging locally to $L_\infty$.

Let $\Omega$ be the class of all nonnegative functions $s(t)$ of $D(\mathbb{R}_1)$ for which

$$\int_{-\infty}^{\infty} s(t) dt = 1.$$  

As usual, by a $\delta$-shaped sequence $\{s_n(t)\}$ at a point $t$ we shall understand a sequence of functions of $\Omega$ with supports which contract to the point $t$.

**Definition.** We say that the number $\gamma(t)$ is a value of the generalized function $f \in D'$ at the point $t$ if there exists a $\delta$-shaped sequence $\{s_n(t)\}$ at the point $t$ such that

$$\lim_{n \to \infty} \langle f, s_n(t) \rangle = \gamma(t).$$  

The upper bound of values of $f$ at the point $t$ is called the upper value of $f$ at the point $t$ and denoted by $\gamma(t)$; the lower bound is called the lower value and denoted by $\alpha(t)$.

The following assertions are readily established:

**Proposition 1.** The quantities $\alpha(t)$ and $\beta(t)$ are values of the generalized function $f \in D'$ at the point $t$. If $\alpha(t)$ and $\beta(t)$ are finite, then the set of values of $f$ at the point $t$ coincide with the interval $[\alpha(t), \beta(t)]$.

**Proposition 2.** A function of the upper (lower) value $\beta(t)$ ($\alpha(t)$) is upper semicontinuous (lower semicontinuous).

**Lemma 1.** An arbitrary function $s \in \Omega$ satisfies the inequality

$$\inf_{\text{supp } s} \alpha(t) \leq \langle f, s \rangle \leq \sup_{\text{supp } s} \beta(t).$$

**Proof.** Assume the contrary to be so. Suppose, for definiteness, that

$$\langle f, s \rangle > \sup_{\text{supp } s} \beta(t) + \varepsilon, \varepsilon > 0.$$  

We cover the $\text{supp } s$ with a finite number of neighborhoods $u_i$. Let $\{\varphi_i\}$ be an expansion of unity corresponding to the covering $\{u_i\}$. Then

$$s(t) = \sum_i s(t) \varphi_i(t) = \sum_i \int_{-\infty}^{\infty} s(t) \varphi_i(t) dt \frac{s(t) \varphi_i(t)}{\int_{-\infty}^{\infty} s(t) \varphi_i(t) dt}.$$  

If we denote $\frac{s(t) \varphi_i(t)}{\int_{-\infty}^{\infty} s(t) \varphi_i(t) dt}$ by $s_i(t)$, then for some function $s_i(t)$ we have

$$\langle f, s_i \rangle > \sup_{\text{supp } s} \beta(t) + \varepsilon,$$

since otherwise we would have

\[ \langle f, s \rangle = \sum_{n=-\infty}^{\infty} s(t)q_n(t)dt \leq (\sup_{\text{supp } s} \beta(t) + \varepsilon) \sum_{n=-\infty}^{\infty} s(t)q_n(t)dt = \sup_{\text{supp } s} \beta(t) + \varepsilon, \]

counter to our assumption. Taking a sequence of refined coverings, we obtain a sequence of functions \( \{s_n\} \) such that \( \langle f, s_n \rangle \leq \sup_{\text{supp } s} \beta(t) + \varepsilon, \) \( \text{supp } s_n \subseteq \text{supp } s \) and the diameter of \( \text{supp } s_n \) tends to zero. From it we may select a \( \delta \)-shaped sequence at some point \( t_0 \in \text{supp } s \). Then \( \beta(t_0) \geq \sup_{\text{supp } s} \beta(t) + \varepsilon, \) which is not possible.

**THEOREM 1.** Let us assume that \( f \in D' \), that \( G \) is an open set, and that \( \alpha = \inf_{G} \alpha(t), \beta = \sup_{G} \beta(t). \) The restriction of \( f \) on the set \( G \) belongs to \( L^\infty(G) \) if and only if \( \alpha \) and \( \beta \) are finite, wherein \( ||f||_{L^\infty(G)} = \max \{||\alpha||, ||\beta||\}. \)

*Proof.* Necessity. For an arbitrary function \( s \in \Omega \) with support in \( G \)

\[ |\langle f, s \rangle| \leq ||f||_{L^\infty(G)} ||s||_{L^1(G)} = ||f||_{L^1(G)}. \]

It follows from this that

\[ |\alpha|, |\beta| \leq ||f||_{L^1(G)}. \quad (1) \]

Sufficiency. We consider the function

\[ \theta_{\alpha}(t) = \begin{cases} ke^{-\frac{t^2}{\alpha^2}}, & |t| \leq \alpha, \\ 0, & |t| > \alpha, \end{cases} \]

where \( k \) is a normalizing factor. Suppose that \( \varphi \in D(G) \). We introduce the function

\[ q_{\alpha} = \varphi * \theta_{\alpha} = \varphi^+ * \theta_{\alpha} - \varphi^- * \theta_{\alpha}, \]

where \( \varphi^+ \) and \( \varphi^- \) are the positive and negative components of \( \varphi \). For a sufficiently small, \( \text{supp } \varphi_{\alpha} \subseteq G \), and

\[ \langle f, \varphi \rangle = \lim_{\alpha \to 0} \langle f, \varphi_{\alpha} \rangle = \lim_{\alpha \to 0} \langle f, \varphi_{\alpha}^+ \rangle - \langle f, \varphi_{\alpha}^- \rangle, \]

where \( \varphi_{\alpha}^+ = \varphi^+ * \theta_{\alpha}, \varphi_{\alpha}^- = \varphi^- * \theta_{\alpha} \). From Lemma 1 it follows that

\[ \alpha \int_{-\infty}^{\infty} q_{\alpha}^+(t)dt \leq \langle f, q_{\alpha}^+ \rangle \leq \beta \int_{-\infty}^{\infty} q_{\alpha}^+(t)dt, \]

\[ -\beta \int_{-\infty}^{\infty} q_{\alpha}^-(t)dt \leq -\langle f, q_{\alpha}^- \rangle \leq -\alpha \int_{-\infty}^{\infty} q_{\alpha}^-(t)dt. \]

Combining the inequalities and passing in them to the limit, we obtain as the final result

\[ |\langle f, \varphi \rangle| \leq \max \{|\alpha|, |\beta|\} \int_{-\infty}^{\infty} |q(t)|dt. \]

Thus the restriction of \( f \) on \( G \) generates a functional continuous in the norm of \( L^1(G) \); consequently, it coincides with a function of \( L^\infty(G) \); moreover

\[ ||f||_{L^\infty(G)} \leq \max \{|\alpha|, |\beta|\}. \quad (2) \]

From inequalities (1) and (2) it follows that \( ||f||_{L^\infty(G)} = \max \{|\alpha|, |\beta|\}. \)

**COROLLARY.** If \( \alpha(t_0) \) and \( \beta(t_0) \) are finite, then in a neighborhood of \( t_0 \) the generalized function belongs to \( L^\infty \). This follows from the fact that in some \( t_0 \) neighborhood \( \alpha(t) \) and \( \beta(t) \) are bounded by virtue of their semicontinuity.

S. Loyasevich (see [1]) gave a definition for the value of a generalized function at a point. It is not hard to see that if a generalized function is such that

\[ \alpha(t_0) = \beta(t_0), \]

then the value of the generalized function at the point \( t_0 \) in the Loyasevich sense is equal to this number. The converse is, in general, not true.

**THEOREM 2.** Suppose that the generalized function \( f \) belongs to \( L^\infty \) locally on an open set \( G \) and let