Let \( q(x) \) be a positive function given on the interval \( I \) of the real axis; let \( P \) be the minimal operator generated in \( L_2(0, +\infty) \) by the differential expression \( P[\cdot] = -d^2/dx^2 + q(x) \); let \( Q \) be the operator of multiplication by the function \( q(x) \). If \( D_P \subseteq D_Q \), then \( P[\cdot] \) is said to be separated. In this note the separation of \( P[\cdot] \) is proved for some growth regularity conditions on the function \( q(x) \), without assuming anything on its smoothness. One proves that if \( D_P \subseteq D_S \), where \( S \) is the multiplication operator by the function \( s(x) \), satisfying some growth regularity condition, then \( D_Q \subseteq D_S \).

1. We consider on the open interval \( I \) of the real axis the differential expression

\[
P[y] = -y'' + q(x)y,
\]

where the function \( q(x) \) is bounded from below by a positive number. Let \( P, \delta_2 \) be the minimal operators generated in \( L_2(I) \) by the differential expressions \( P[\cdot] \) and \( -d^2/dx^2 \), respectively. We denote the operators conjugate to \( P, \delta_2 \), by \( P^*, \delta_2^* \). Let \( Q \) be the operator of multiplication by the function \( q(x) \). We denote by \( D_Q, D_\delta, D_\delta^* \) the domains of the definition of the operators \( Q, P^*, \delta_2^* \).

**Definition 1.** The differential expression \( P[\cdot] \) is said to be separated if the inclusion

\[
D_P \subseteq D_Q.
\]

holds.

In [1-4] and in other papers, W. N. Everitt and M. Giertz have considered the separation problem for Sturm-Liouville operators and for their powers and they have obtained a series of fundamental results, one of the most important being the following theorem:

Let \( q \) be locally integrable with its square on \([0, +\infty)\) and assume that for some \( k, x_0 > 0 \) it satisfies the conditions

1. \( q(x) \geq k > 0 \ (x \in [x_0, +\infty)) \),
2. \( q'(x) \) is absolutely continuous on \([x_0, X]\) for all \( X > x_0 \),
3. \( g^{-1}(q^{-1}g) \subseteq L_1(x_0, +\infty) \),

then \( P[\cdot] \) is separated (see [1]).

In the same paper one proves, under some additional conditions on the function \( q(x) \), that if

\[
D_P \subseteq D_{q_{1+\alpha}},
\]

where \( \alpha \geq 0 \), then \( \alpha = 0 \).

In [2], the separability of the differential expression \( P[\cdot] \) on the interval \( I \) is obtained in the case when \( q(x) \) is absolutely continuous on each closed interval \([x_1, x_2]\), for which \( (x_1, x_2) \subseteq I \) and under the assumption that one of the following conditions holds:

1. \( I \) is a finite interval with regular end-points,
At one end-point of the interval I we have the case of a limit point and for all \( x \in I \) and for some positive \( \varepsilon < 1 \) we have the inequality
\[
\left| q'(x) \right| \leq (1 - \varepsilon) \left| q(x) \right|,
\]

(3) \( I = (a, + \infty) \), where \( a > -\infty \) and \( q(x) \) is bounded on \((a, a + \delta)\) for some \( \delta > 0 \).

2. Definition 2. The real function \( q(x) \) is said to be a function of class \( R^6(I) \) (\( \varepsilon, \delta \) are positive numbers), if \( q(x) \) is bounded on each bounded subset of the interval \( I \) and for \( x, y \in I \), sufficiently large in absolute values, and satisfying the inequality \( \delta \mid x - y \mid \|q(y)\| \leq 1 \), we have the inequality
\[
\left| q(x) - q(y) \right| \leq \varepsilon \|q(y)\|.
\]

In the present paper we obtain the separation of the differential expression \( P[\cdot] \) without any smoothness condition on the potential \( q(x) \).

3. Let \( \psi \in C^\infty_0(R) \) and assume that we have the inequalities
\[
0 \leq \psi(x) \leq 1, \quad (x \in R_1); \quad \psi(x) = 1, \quad (\mid x \mid < 4/3); \quad \psi(x) = 0, \quad (\mid x \mid > 1);
\]
we denote: \( k_1 = \sup \mid \psi'(x) \mid, \quad k_2 = \sup \mid \psi''(x) \mid \).

We have the following:

Theorem 1. Let \( q(x) \) be a measurable function of the class \( R^6(I) \), where \( \delta, \varepsilon \) are positive numbers satisfying the inequality
\[
(\varepsilon + 2k_1\delta + k_2\delta^2) \leq (1 + \varepsilon)^{1/4}(1 - \varepsilon)^{1/4},
\]
then \( P^* = \partial_2^* + Q\).

We note that \( P^* = \partial_2^* + Q \) does not hold for every function \( q(x) \).

For example, for the function
\[
q_1(x) = \begin{cases} 
\exp n^3 & (\mid x - n \mid \leq 0, 5n^{-2}\exp(-n^2)), \\
0 & \text{at the remaining points},
\end{cases} 
\]
\( n = 1, 2, 3, \ldots \)

it is known (see [1], p. 317) that the differential expression
\[
P_1[y] = -y'' + q_1(x)y \quad (0 < x < +\infty),
\]
is not separated, i.e., \( D_{P_1^*} \not\subset D_{Q_1} \), consequently, \( P_1^* = \partial_2^* + Q_1 \) since otherwise we would have the equality \( D_{P_1^*} = D_{\partial_2^*} \cap D_{Q_1} \).

In the sequel we will consider only such functions \( q(x) \) for which the conditions of Theorem 1 hold.

**Lemma 1.** The domain of definition \( D_{P^*} \) of the operator \( P^* \) consists of the functions \( f \in L_2(I) \), for which \( f \) is absolutely continuous on each compact subset of the interval \( I \) and the function
\[
-f''(x) + q(x)f(x)
\]
belongs to \( L_2(I) \).

This proposition is a consequence of the results of §17 of [5] (Russian 1969 edition).

**Lemma 2.** Assume that the differential expression
\[
P[y] = -y'' + q(x)y \quad (x \in R_1)
\]
is separated. We denote by \( \tilde{q}(x) \) the restriction of the function \( q(x) \) to the interval \( I \).

Then the differential expression
\[
P[y] = -y'' + \tilde{q}(x)y \quad (x \in I)
\]
is separated.

**Proof.** Let \( f \in D_{\tilde{P}^*} \). There exists a function \( \varphi \in C^\infty_0(I) \), for which \( \varphi(x) = 1 \), if the distance from the point \( x \) to the endpoints of the interval \( I \) is larger than 1. From Lemma 1 and from the boundedness of the function \( q(x) \) on the set
\[
\text{supp}(1 - \varphi) = \{x \mid x \in I, \varphi(x) \neq 1\},
\]

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