A NEW CONCEPT OF PREDICATIVE TRUTH
AND DEFINABILITY

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In this paper a conception of predicative truth in the language of naive analysis is given, and the peculiarities of the induced logic are studied. It is shown that this concept can be formalized in a system with a constructive Carnap rule and that in a theory based on this concept of truth the same things can be expressed that can be expressed in the usual theories of hyperarithmetic and ramified analysis.

If one does not consider a set as a basic object of a mathematical theory but believes a set is only a useful abstraction which serves as a convenience for formulation and expansion of the language of the theory, then it is natural to require the following properties for all sets used and for the notion of truth:

1. Nameability, i.e., unique definability by a formula of the theory, using the primitive objects of the theory as constants.

2. The truth of more complicated formulas must be uniquely determined by the truth of simpler formulas.

3. Formulas which do not use the concept of sets must be true in the original theory iff they are true in the augmented theory (consistency). If we define truth in the original theory by deducibility, then in the augmented theory it must be defined by deducibility preserving consistency.

4. Consistency of truth in the augmented theory with the rules of standard logic; i.e., $A \& B$ is true iff $A$ and $B$ are true; if $A$ (or $B$) is false, then $A \& B$ is false. $\neg A$ is true iff $A$ is false, etc.

Up until now, the known method for such augmenting of a theory was in ramified analysis (see [1]). However, this theory is somewhat inconvenient, in that all sets are divided into layers, and one can say "for all sets" or "there is a set" only in the metalanguage. In this paper another method is suggested for a similar augmentation of arithmetic, which does not have the above-mentioned deficiencies and satisfies all four requirements. In distinction from the usual theories of hyperarithmetic (see [1]) or ramified analysis our theory uses unbounded rules of "deduction" and an overdetermined notion of truth for sets in a predicative and noncontradictory way. However, classical logic in the full system is preserved only for certain sets.

§ 1. Formulation of the Language and the Definition of Truth

As the basic language we take the language of naive analysis (An). The language is constructed in such a way that in it one can define the sets of all known and naive set theories, including those which lead to paradoxes.

1. Symbols of the language:

0, 1, 2, . . . — symbols for natural numbers.

$x, y, z, . . .$ — variables for natural numbers.
\( f^n, g^n, h^n, \ldots \) - symbols for \( n \)-places primitive recursive functions (PRF).

\( = \) - equality.

\( \Rightarrow, \forall \) - logical symbols.

\( \in, \exists \) - special symbols.

Round parenthesis, Comma.

We choose some Gödel numbering of the words on these symbols.

Terms:

1. Natural numbers and variables are terms.

2. If \( f^n \) is an \( n \)-place PRF, \( t_1, t_2, \ldots, t_n \) are terms, then \( f^n(t_1, t_2, \ldots, t_n) \) is a term.

We assume that to each function symbol uniquely corresponds a PRF and we can find the symbol for each PRF. For terms which do not contain variables we may in a natural way define its value \( \| \cdot \| \).

Formulas:

1. If \( t \) and \( u \) are terms, then \( (t = u) \) is an elementary formula.

2. If \( A \) and \( B \) are formulas, then \( (A \Rightarrow B) \) is a formula.

3. If \( A \) is a formula and \( x \) is a variable, then \( \forall x A \) is a formula.

4. If \( A \) is a formula, \( x \) is a variable, and \( t \) is a term, then \( (t \in \forall x A) \) is a formula.

5. If \( t \) and \( u \) are terms, then \( (t \in u) \) is a formula.

Objects of the form \( \forall x A \), where \( A \) is a formula and \( x \) is a variable, we will call predicators.

The closed predicator with Gödel number \( n \) will be denoted \( M_n \). The notion of substitution of a term for a variable in a formula, term, or predicator is defined in the usual way.

In order to define the concepts of true and false formulas we first construct an operator \( \rho \) which in certain circumstances will enable us to understand the meaning of formulas obtained by rules 4 and 5. This operator will be applied only to closed formulas of such a form and will yield closed formulas of \( \Delta_n \).

1. \( \rho_\perp (t \in \forall x A) \perp A (x \mid t) \).

2. \( \rho_\perp (t \in u) \perp (t \in M_n) \), where \( n \) is the value of \( u \).

The set \( X \) will be called \( \epsilon \)-closed if \( X \) is a set of pairs of the form \( (A, T) \) or \( (A, F) \), where \( A \) is a formula of \( \Delta_n \) and

a) If \( t \) and \( u \) are closed terms then if \( \| t \| = \| u \| \) then \( (t = u), T \in X \), and if \( \| t \| \neq \| u \| \) then \( (t = u), F \in X \).

b) If \( \langle A, F \rangle \in X \) (or \( \langle B, T \rangle \in X \)), then \( \langle A \Rightarrow B, T \rangle \in X \).

c) If \( \langle A, T \rangle \in X \) and \( \langle B, F \rangle \in X \), then \( \langle A \Rightarrow B, F \rangle \in X \).

d) If for each \( n \), \( \langle A(x \mid n), T \rangle \in X \), then \( \langle \forall z A, T \rangle \in X \).

e) If for some \( t \), \( \langle A(x \mid t), F \rangle \in X \), then \( \langle \forall z A, F \rangle \in X \).

f) If \( \rho_{\perp A \perp}, T \rangle \in X \) (or \( \rho_\perp A \perp, F \rangle \in X \)), then \( \langle A, T \rangle \in X \) (or \( \langle A, F \rangle \in X \)).

The smallest \( \epsilon \)-closed set is denoted by \( S \). If \( \langle A, T \rangle \in S \), we will say that \( A \) is true, and if \( \langle A, F \rangle \in S \), we will say that \( A \) is false. Only closed formulas may be either true or false.

In order to examine the concept of truth more easily we will define the true and false formulas inductively. We consider the ordinal semantics \( S_\alpha \).

a) If \( A \) is a true elementary closed formula, then \( \langle A, T \rangle \in S_\alpha \); if \( A \) is a false elementary closed formula then \( \langle A, F \rangle \in S_\alpha \).