CHANGE OF VARIABLE IN THE ONE-DIMENSIONAL
LEBESGUE INTEGRAL

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We obtain necessary and sufficient conditions for the change of the variable in a one-dimen-
sional Lebesgue integral in the case when the substituting function is quasibijective (see
Definition 1).

Definition 1. A function \( \varphi(x) \), given on the measurable set \( X \subseteq \mathbb{R} \), will be said to be quasibijective on
\( X \), if there exists a measurable \( W \subseteq X \) such that:

1) \( \varphi(x) \) is bijective on \( W \);
2) \( m\varphi(X \setminus W) = 0 \).

Lemma 1. Let \( g(s) : [a, b] \rightarrow \mathbb{R} \) and \( g(s) \in \text{Lip}(1) \). Assume that \( B \subseteq [a, b] \) is a closed set and that
\( g(s) \) is quasibijective on \( B \). Then, there exists a function \( \widetilde{g}(s) \) on \( [a, b] \), such that:

1) \( \widetilde{g}(s) \) is bounded and almost everywhere differentiable;
2) \( \widetilde{g}(s) \) is quasibijective on \( [a, b] \);
3) \( \widetilde{g}(s) = g(s) \) for every \( s \in B \);
4) \( \widetilde{g}'(s) = g'(s) \) almost everywhere on \( B \);
5) \( \widetilde{g}(s) \) possesses property (N) on \( [a, b] \); i.e., \( \widetilde{g}(s) \) maps sets of measure 0 on \( [a, b] \) into sets of measure 0.

Proof. We will consider, for the sake of simplicity, that \( a \in B \) and \( b \in B \). Then \( U = [a, b] \setminus B \) is open.
Let \( U = \bigcup (a_n, b_n) \), where \( (a_n, b_n) \) are nonintersecting intervals. Consequently, \( a_n, b_n \in B \) for every \( n \).
We denote \( s_n = (a_n + b_n)/2 \), and we divide the segment \( (a_n, s_n) \) into the segments \( \Delta_{n,k} = (x_{n,k+1}, x_{n,k}) \), where
\( k = 0, 1, 2, \ldots \), choosing the points \( x_{n,k} \) in the following manner:

\[
x_{n,k} = a_n + \frac{(s_n - a_n)}{[1 + (s_n - a_n)]^k} \quad \text{for} \quad k = 0, 1, 2, \ldots .
\]

Consequently, \( x_n = x_{n,0} > x_{n,1} > \ldots > x_{n,k} > \ldots \). From (1) we can also see that \( x_{n,k} \rightarrow a_n \) for \( k \rightarrow \infty \). Then
\( \{x_{n,k}\} = (a_n, s_n) \) for every \( n \). We denote by \( |\Delta_{n,k}| = x_{n,k} - x_{n,k+1} \) the length of the segment \( \Delta_{n,k} \). Then

\[
|\Delta_{n,k}| = \frac{b_n - a_n}{2} \cdot (x_{n,k+1} - a_n) \leq \frac{b_n - a_n}{2} \cdot d(x_{n,k+1}, B),
\]

where \( d(x, B) \) denotes the distance from the point \( x \) to the set \( B \). Similarly, we divide the right half \( (s_n, b_n) \)
into the segments \( \Delta^+_{n,k} = (x^+_{n,k}, x^+_{n,k+1}) \), where

\[
x^+_{n,k} = b_n - \frac{(b_n - s_n)}{[1 + (b_n - s_n)]^k} ;
\]

in this case \( \{x^+_{n,k}\} (k = 0, 1, 2, \ldots) \) is a monotonically increasing sequence \( x^+_{n,0} = s_n \) and \( x^+_{n,k} \rightarrow b_n \) for
\( k \rightarrow \infty \). One can obtain an estimate similar to the estimate (2):

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Every interval \((a_n, b_n)\) from \(U\) breaks up now into the segments
\[
(a_n, b_n) = \left( \bigcup_{k=0}^{\infty} \Delta_{n,k}^{-} \right) \cup \left( \bigcup_{k=0}^{\infty} \Delta_{n,k}^{+} \right).
\]

We define the function \(\overline{g}(s)\) on \([a, b]\) = \(U \cup B\), by setting:
\[
1) \overline{g}(s) = g(s) \quad \text{for} \quad s \in B;
2) \overline{g}(s) = g(s_{n,k}) \quad \text{for} \quad s \in \Delta_{n,k}^{-};
3) \overline{g}(s) = g(s_{n,k+1}) \quad \text{for} \quad s \in \Delta_{n,k}^{+}.
\]

One can see from (4) that the formulas (5) define the function \(\overline{g}(s)\) on the entire \([a, b]\). Let us prove that the so constructed function \(\overline{g}(s)\) satisfies the conditions 1-5) of Lemma 1.

1. Since by construction \(\overline{g}([a, b]) \subset g([a, b])\), then \(g(s)\) is bounded together with \(g(s)\). Further, \(\overline{g}(s)\) is, by construction, a step function on \(U\); consequently, it is almost everywhere differentiable and possesses property (N) on \(U\). Since on \(B\) we have \(\overline{g}(s) \equiv g(s)\) by construction, it follows that \(\overline{g}(s)\) possesses property (N) also on \([a, b]\). Let us prove now that if \(s_0 \in B\), \(s_0 \neq a_n, b_n\) and \(g'(s_0)\) exists, then \(\overline{g}'(s_0)\) exists and \(\overline{g}'(s_0) = g'(s_0)\).

It is sufficient to prove that
\[
\lim_{s \to s_0} \left| \frac{\overline{g}(s) - \overline{g}(s_0)}{s - s_0} - \frac{g(s) - g(s_0)}{s - s_0} \right| = 0 \quad \text{for} \quad s \to s_0.
\]

Since for \(s \in B\) the last difference is equal to zero, we may consider that \(s \in U\). Then \(s \in \Delta_{n,k}^{-}\) or \(s \in \Delta_{n,k}^{+}\) for some \(n = n(s)\) and \(k = k(s)\). From the conditions \(s_0 \neq a_n, b_n\) it follows that \(s_0\) is a two-sided limit point of \(B\). We may have two cases:

a) \(s_0 \in B\). Then, in some neighborhood of \(s_0\) we have \(\overline{g}(s) \equiv g(s)\) and (6) is trivially satisfied.

b) \(s_0 \not\in B\). In this case, obviously \(n(s) \to \infty\) for \(s \to s_0\) and \(s \in U\); \((s \in \Delta_{n,k}^{+})\).

Let us estimate now the difference (6):
\[
\frac{\overline{g}(s) - \overline{g}(s_0)}{s - s_0} - \frac{g(s) - g(s_0)}{s - s_0} = \left| \frac{\overline{g}(s) - g(s)}{s - s_0} - \frac{\overline{g}(s) - g(s)}{s - s_0} \right| = \left| \frac{\overline{g}(s) - g(s)}{s - s_0} \right|,
\]
since \(\overline{g}(s_0) = g(s_0)\). Then, for \(s \in \Delta_{n,k}^{+}\), making use of (5), we obtain
\[
\left| \frac{\overline{g}(s) - g(s)}{s - s_0} \right| = \left| \frac{g(s) - g(s_{n,k+1})}{s - s_0} \right|.
\]

Making use of the fact that \(g(s) \in \text{Lip}(1)\) (with some constant \(C\)) and of the conditions (2) and (3), we obtain
\[
\left| \frac{g(s) - g(s_{n,k+1})}{s - s_0} \right| \leq \frac{C}{|s - s_0|} \left| \frac{\Delta_{n,k}}{|s - s_0|} \right| \leq \frac{C b_n - a_n}{2} \frac{d(s_{n,k+1}, B)}{|s - s_0|} \leq C \frac{b_n - a_n}{2} \frac{d(s_{n,k+1}, B)}{|s - s_0|} \leq \frac{C b_n - a_n}{2},
\]

since \(d(s, B) \leq |s - s_0|\). As remarked earlier, \(n(s) \to \infty\) for \(s \to s_0\). Consequently, \((b_n(s) - a_n(s)) \to 0\) for \(s \to s_0\). Therefore, from (7) we obtain
\[
\left| \frac{\overline{g}(s) - g(s)}{s - s_0} - \frac{g(s) - g(s_0)}{s - s_0} \right| \leq \frac{C b_n(s) - a_n(s)}{2} \to 0 \quad \text{for} \quad s \to s_0.
\]

2. Let us prove that \(\overline{g}(s)\) is quasibijective on \([a, b]\). Since \(g(s)\) is quasibijective on \(B\), there exists, by definition, \(W \subset B\) such that \(g(s)\) is bijective on \(W\) and \(m(g(B\setminus W)) = 0\). Since \(\overline{g}(s)\) coincides with \(g(s)\) on \(B\), \(\overline{g}(s)\) is also bijective on \(W\). In addition
\[
m(\overline{g}(B\setminus W)) = m(g(B\setminus W)) + m(g(U)) = m(g(B\setminus W)) + m(g(U)) = 0.
\]