LINEARLY ORDERED GROUPS WHOSE SYSTEM OF CONVEX SUBGROUPS IS CENTRAL

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The order $P$ on a group $G$ is called rigid if for $p \in P$ the relation $p[x, p] \in P$ holds for every $x \in G$, $e = \pm 1$. In this note we give criteria for the existence of a rigid linear order, for the extendability of a rigid partial order to a rigid linear order, and for the extendability of each rigid partial order to a rigid linear order on a group. It is proved that the class of groups each of whose rigid partial orders can be extended to a rigid linear order is closed with respect to direct products. A new proof of the theorem of M. I. Kargapolov which states that if a group $G$ can be approximated by finite $p$-groups for infinite number of primes $p$, then it has a central system of subgroups with torsion-free factors is presented.

Let us recall that a group $G$ has a central system of subgroups if it has a complete invariant system $\Sigma$ of subgroups $G \supseteq \ldots \supseteq G_2 \supseteq G_1 \supseteq \ldots \supseteq E$ containing $G$ and the trivial group $E$ such that $[G, G]_G \subseteq G_\alpha$ for every jump $G_\alpha \supseteq G_\beta$ of subgroups of $\Sigma$. We know that the class of groups having a central system can be universally axiomatized and is, moreover, a quasivariety. In the study of linear and partial orders on nilpotent groups it has been observed (see [1] and also [2]) that the system of convex subgroups of a nilpotent group is central. This fact is an immediate consequence of the fact that the inequality

$$|a, b| = \min \{ |a|, |b| \}$$

is valid for arbitrary positive elements $a$ and $b$ in every linearly ordered nilpotent group. We will say that a partial order $P$ on a group $G$ is rigid if for arbitrary $p \in P$ we have $p[x, p] \in P$, $\forall e \subseteq G, e = \pm 1$. It follows immediately from the Mal'tsev-Podderyugin-Riger criterion for the orderability of a group that a group $G$ admits a rigid linear order if and only if it has a central system of subgroups whose factors are torsion-free. Lattice-ordered groups with rigid orders have been considered in [3]. It is shown there that this class of groups forms a subvariety of the variety of lattice-ordered groups (with respect to the group operations and the operations of union and intersection) and is given by the identity $(x \lor y)^2 \lor (y^{-1} (x \lor e)y)$ $(x \lor y)^2$, and also that every rigid lattice-order on a group can be extended to a rigid linear order.

In this note we give in the language of invariant subgroups criteria for the existence of a rigid linear order on a group, for the extendability of a rigid partial order on a group to a rigid linear order, and for each rigid partial order on a group to be extendable to a rigid linear order. In the second section, it is proved that the class of groups, each of whose rigid partial orders can be extended to a rigid linear order, is closed with respect to direct products and is also given a proof of the theorem of M. I. Kargapolov which states that if a group $G$ can be approximated by finite $p$-groups for infinite number of prime numbers $p$, then it has a central system of subgroups with torsion-free factors. As usual, we will denote by $S(a_1, \ldots, a_n)$ the smallest invariant subgroup of the group $G$ containing the elements $a_1, \ldots, a_n$. By $P(a_1, \ldots, a_n)$ we will denote the smallest invariant subgroup of the group $G$ containing $a_1, \ldots, a_n$ and together with every element $g \in P(a_1, \ldots, a_n)$ every element of the form $g[x, g]^y$ for arbitrary $x, y \in G$.

§1. We will say that an invariant subgroup $P$ of a group $G$ has the property (no) if for an arbitrary finite set of elements $a_1, \ldots, a_n, x_1, \ldots, x_n$ of the group $G$, $a_i \neq e_j, i = 1, \ldots, n$, there exist $e_1, \ldots, e_n$ equal to $\pm 1$ such that $P \not\models S(a_1^e, x_1, a_1, \ldots, a_n^e, x_n, a_n)$. \(\Phi\)
LEMMA I. If an invariant subgroup $P$ of a group $G$ has the property (no), then for arbitrary $a \in G$, $a \neq e$, $PS(a^e, e)$ also has the property (no) for a suitable choice of the sign $\varepsilon = \pm 1$.

Indeed, if $PS(a, e)$ and $PS(a^{-1}, e)$ do not have the property (no), then there exist elements $u_1, \ldots, u_n; x_1, \ldots, x_n$ and $v_1, \ldots, v_m; y_1, \ldots, y_m; u_i \neq e, v_j \neq e$ such that

$$PS(a, e) \cap S(u_1 \{x, u_1\}, \ldots, u_n \{x, u_n\}) \neq \phi,$$

$$PS(a^{-1}, e) \cap S(v_1 \{y, v_1\}, \ldots, v_m \{y, v_m\}) \neq \phi$$

for arbitrary sets of signs $\varepsilon_i, \eta_j$, equal to $\pm 1$. This means that

$$P \cap S(a^{-1}, u_1 \{x, u_1\}, \ldots, u_n \{x, u_n\}) \neq \phi,$$

$$P \cap S(a, v_1 \{y, v_1\}, \ldots, v_m \{y, v_m\}) \neq \phi$$

for arbitrary $\varepsilon_i, \eta_j = \pm 1$. The latter relations contradict the fact that $P$ has the property (no) since for elements $a_1, u_1, \ldots, u_n; v_1, \ldots, v_m; x_1, \ldots, x_n; y_1, \ldots, y_m$ (where $a_1 \neq e; u_1 \neq e; v_1 \neq e$) we have

$$P \cap S(a_1 \{x, a_1\}, \ldots, u_n \{x, a_1\}; v_1 \{y, v_1\}, \ldots, v_m \{y, v_m\}) \neq \phi$$

for an arbitrary choice of the signs $\varepsilon_i, \eta_j; \varepsilon_i = 1, \ldots, n; \eta_j = 1, \ldots, m$.

THEOREM 1. A partial order $P$ on a group $G$ can be extended to a rigid linear order if and only if $P$ has the property (no).

Let $P$ be extendable to a rigid linear order $Q$ on the group $G$ and $a_1, \ldots, a_n; x_1, \ldots, x_n$ be a set of elements of the group $G$. Let us choose the signs $\varepsilon_i$ such that $a_1^{-1} \leq e$. Since $Q$ is a rigid order, it follows that $Q \cap S(a_1 \{x, a_1\}, \ldots, a_n \{x, a_n\}) = \phi$. Whence, it follows that $P$ has the property (no).

Conversely, if $P$ has the property (no), then we imbed $P$ in a maximal order $Q$ having the property (no). Let $a \in G$. Then $QS(a, e)$ or $Q(a^{-1}, e)$ is a partial order having the property (no), and, by virtue of the maximality of $Q$, we have $QS(a, e)$ or $QS(a^{-1}, e) = Q$; whence it follows that $a \in Q$ or $a^{-1} \in Q$.

COROLLARY 1. A group $G$ has a central system of subgroups with torsion-free factors if and only if for $a_1, \ldots, a_n; x_1, \ldots, x_n$ of the group $G$ such that $a_1 \neq e$ there exists a set of signs $\varepsilon_1, \ldots, \varepsilon_n$ equal to $\pm 1$ such that

$$S(a_1 \{x, a_1\}, \ldots, a_n \{x, a_n\}) \equiv e.$$

COROLLARY 2. The class of groups having a central system of subgroups with torsion-free factors is closed with respect to Cartesian products, taking of subgroups, is locally closed, and is consequently a quasivariety.

The following theorem gives necessary and sufficient conditions for the extendability of an arbitrary rigid partial order to a rigid linear order.

THEOREM 2. Every rigid partial order on a group $G$ can be extended to a rigid linear order on $G$ if and only if

1. $e \in P(x)$ for arbitrary $x \in G$ such that $x \neq e$,
2. the relation $P(y) \cap P(z) = \phi$ holds for arbitrary $x, y, z \in G$ such that $y, z \in P(x)$.

Let every rigid partial order on the group $G$ be extendable to a rigid linear order on $G$. Then it is obvious that (1) is fulfilled in $G$. Let us assume that (2) does not hold, i.e., let there exist $x, y, z \in G$ such that $y, z \in P(x)$ and $P(y) \cap P(z) = \phi$. Then $P(y, e)P(z^{-1}, e) = T$ is a rigid partial order. Indeed, as we know (see [2, §1, Chap. III]), $T$ is a partial order. But at the same time it is rigid since if $a \in P(y, e)P(z^{-1}, e)$, then $a = y_1z_1$, where $y_1 \in P(y, e)$, $z_1 \in P(z^{-1}, e)$, and

$$a \{b, a\} = y_1z_1 \{b, y_1z_1\} = y_1z_1 \{b, z_1\} \{b, y_1\} = (y_1z_1 \{b, z_1\}y_1^{-1})(y_1 \{x, y_1^{-1}\}),$$

and consequently $[a, b] \in T$.

However, the rigid partial order $T$ cannot be extended to a rigid linear order since if $x > e$, then, as is easily seen, $x[a, x] > e$ and $x[a, x]^{b} > e$ for arbitrary $a, b \in G$. But then the relations $y_1 > e$ and $z^{-1} > e$ contradict each other. The case $x < e$ is analogously excluded.