Some models of combinatory logic

D. Skordev

Some models for the weak theory of combinators are described. These models consist of partial functions having natural number arguments and natural number values. The role of the application operation is played by certain \( \mu \)-recursive operators.

In this note, by \( n \)-place function we shall mean a partial function of \( n \) natural number arguments which takes natural number values. If \( m_1, \ldots, m_p \) and \( n \) are natural numbers, then by an operator of type \( m_1, \ldots, m_p \rightarrow n \) we shall mean any mapping which corresponds an \( n \)-place function to each system \( \langle \xi_1, \ldots, \xi_p \rangle \), where each \( \xi_i \) is an \( m_i \)-place function \((i = 1, \ldots, p)\).

**Theorem.** Let there be given natural numbers \( a, k, \) and \( s \), with \( k \leq s \). Then one may construct a \( \mu \)-recursive operator* \( \Phi \) of type \( 1, 1 \rightarrow 1 \), such that, for every choice of \( 1 \)-place functions \( \nu \) and \( \sigma \), satisfying the conditions \( \nu(a) = k \) and \( \sigma(a) = s \), the following two equations hold for all \( 1 \)-place functions \( \xi, \eta, \zeta \):

\[
\Phi (\Phi (\nu, \xi), \eta) = \xi, \quad \Phi (\Phi (\sigma, \xi), \eta, \zeta) = \Phi (\Phi (\xi, \zeta), \Phi (\eta, \zeta)).
\]

**Proof.** First, let us construct a \( \mu \)-recursive operator \( \Psi \) of type \( 1, 1 \rightarrow 2 \) with the following property: For every \( \mu \)-recursive operator \( F \) of type \( 1, 1 \rightarrow 1 \) there is a natural number \( c \) such that for all \( 1 \)-place functions \( \xi \) and \( \eta \) there holds the equation

\[
F (\xi, \eta) = \lambda t [\Psi (\xi, \eta) (c, t)].
\]

Such an operator may be constructed by considering a suitable enumeration of the set of \( \mu \)-recursive operators. In fact this operator may be constructed with the following additional properties. By Lemma 3 of [2] we may construct a \( \mu \)-recursive operator \( \Psi_0 \) of type \( 1 \rightarrow 2 \) such that every \( \mu \)-recursive operator \( G \) of type \( 1 \rightarrow 1 \) is representable in the form

\[
G (\xi) = \lambda t [\Psi_0 (\xi) (c, t)],
\]

where \( c \) is a suitably chosen natural number. We set

\[
\Psi (\xi, \eta) = \Psi_0 (\xi),
\]

where \( \xi \) is defined by the equations

\[
\xi (2t) \simeq \xi (t), \quad \xi (2t + 1) \simeq \eta (t).
\]

We then define the operator \( \Phi_0 \) of type \( 1, 1 \rightarrow 1 \) by the equation

\[
\Phi_0 (\xi, \eta) = \lambda t [\Psi (\xi, \eta) (\xi (a), t)].
\]

Clearly, the operator \( \Phi_0 \) is \( \mu \)-recursive. Let us now prove two additional assertions.

1. Let \( G \) be any \( \mu \)-recursive operator of type \( 1 \rightarrow 1 \). Then one can find a natural number \( c \) such that for every choice of \( 1 \)-place function \( \xi \) the condition \( \xi (a) = c \), and for every \( 1 \)-place function \( \eta \) there holds the equation

\[
G (\eta) = \Phi_0 (\xi, \eta).
\]

*Apropos a definition of the concept of \( \mu \)-recursive operator see, e.g., [1].


©1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.
Indeed, it suffices to choose \( c \) with the property that for all 1-place functions \( \xi \) and \( \eta \) there holds

\[
G (\eta) = \lambda t \ [\Psi (\xi, \eta) (c, t)].
\]

2. Let \( p \) be a positive natural number and let \( G \) be a \( \mu \)-recursive operator of type \( 1, \ldots, 1 \rightarrow 1 \).

Then there is a \( \mu \)-recursive operator \( H \) of type \( 1, \ldots, 1 \rightarrow 1 \) such that for all 1-place functions \( \xi_1, \ldots, \xi_p \) and \( \eta \) there holds the equation

\[
G (\xi_1, \ldots, \xi_p, \eta) = \Phi_0 (H (\xi_1, \ldots, \xi_p), \eta).
\]

For the proof of this assertion we consider the operator \( F \) of type \( 1, 1 \rightarrow 1 \) which is defined by the following equation

\[
F (\xi, \eta) = G (\lambda t \ [\xi (a + 1 + pt)], \lambda t \ [\xi (a + 2 + pt)], \ldots, \lambda t \ [\xi (a + p + pt)], \eta).
\]

The operator \( F \) is \( \mu \)-recursive and, consequently, there exists a natural number \( c \) such that for all 1-place functions \( \xi \) and \( \eta \) there holds the equation

\[
F (\xi, \eta) = \lambda t \ [\Psi (\xi, \eta) (c, t)].
\]

Once such \( c \) has been found we may set

\[
H (\xi_1, \ldots, \xi_p) = \xi,
\]

where \( \xi (x) = c \) for \( x \leq a \) and

\[
\xi (a + i + pt) \simeq \xi_i (t)
\]

for \( i = 1, 2, \ldots, p \).

Let us now apply assertion 2 to the operator \( G (\xi, \eta) = \xi \). We obtain a \( \mu \)-recursive operator \( H \) of type \( 1 \rightarrow 1 \) such that for all 1-place functions \( \xi \) and \( \eta \) there holds the equation

\[
\Phi_0 (H (\xi), \eta) = \xi.
\]

Applying assertion 1 to the operator \( H \) we find a number \( k_0 \) such that for every choice of 1-place function \( \kappa_0 \), which satisfies the condition \( \kappa_0 (a) = k_0 \), and for all 1-place functions \( \xi \) and \( \eta \) there holds the equation

\[
\Phi_0 (\Phi_0 (\kappa_0, \xi), \eta) = \xi.
\]

If we apply assertion 2 to the operator

\[
G (\xi, \eta, \xi) = \Phi_0 (\Phi_0 (\xi, \xi), \Phi_0 (\eta, \xi)),
\]

we obtain a \( \mu \)-recursive operator \( G_1 \) of type \( 1, 1 \rightarrow 1 \) with the following property: for all 1-place functions \( \xi, \eta \), and \( \xi \) there holds the equation

\[
\Phi_0 (G_1 (\xi, \eta), \xi) = \Phi_0 (\Phi_0 (\xi, \xi), \Phi_0 (\eta, \xi)).
\]

If in turn we apply assertion 2 to the operator \( G_1 \) and then apply assertion 2 to the resulting corresponding operator \( H \), we obtain a natural number \( s_0 \) such that for every choice of 1-place function \( \sigma_0 \) satisfying the condition

\[
\sigma_0 (a) = s_0
\]

and for all 1-place functions \( \xi, \eta \), and \( \xi \) there holds the equation

\[
\Phi_0 (\Phi_0 (\Phi_0 (\sigma_0, \xi), \eta), \xi) = \Phi_0 (\Phi_0 (\xi, \xi), \Phi_0 (\eta, \xi)).
\]

If 1-place functions \( \kappa_0 \) and \( \sigma_0 \) are such that Eqs. (1) and (2) hold for all 1-place functions \( \xi, \eta \), and \( \xi \), then the equation \( \kappa_0 = \sigma_0 \) is impossible. Indeed, suppose that \( \kappa_0 = \sigma_0 \).

Then

\[
\Phi_0 (\Phi_0 (\Phi_0 (\kappa_0, \kappa_0), \kappa_0), \kappa_0) = \Phi_0 (\Phi_0 (\Phi_0 (\sigma_0, \kappa_0), \kappa_0), \kappa_0),
\]

which gives us