In this article a special diagram is constructed which establishes a connection between different obstructions to surgery and obstructions to decomposition. The diagram enables us to calculate the Wall groups \( L_n(Z_2 \oplus Z_2, \phi) \), and also to obtain some information about homomorphisms into the exact sequence of Sullivan. The bibliography contains five references.

1.1. Notation. \( D = D^2 \) is the two dimensional disk, \( E \subseteq S^1 = \partial D \subseteq D \) is an interval on the circumference, \( E' \subseteq E \) is a smaller interval. Let \( X \) be a manifold (smooth or piecewise linear, with a boundary or without), and let \( Y \) be a submanifold, where \( \partial Y \subseteq \partial X \). We shall denote "\( X \) cut along \( Y \)" by \( X_Y \), that is, \( X \setminus U \) where \( U \) is a tubular neighborhood of \( Y \). We have \( \partial (X_Y) = \partial X \cup \partial U \).

Let \( X' \) be a manifold, \( Y' \subseteq X' \) be a submanifold, and \( X = X' \times D, Y = Y' \times D \).

Let us give the definition of the "group of obstructions to the decomposition of the smoothings of \( X \) along \( Y \)". Let us consider the set of simple, homotopic equivalences \( f : (M, \partial M) \rightarrow (X, \partial X) \) such that the following conditions are satisfied:

- \( f|_{(X' \times \partial D)} : (X' \times E) \rightarrow (X' \times E) \) is the diffeomorphism,
- \( f|_{(Y' \times \partial D)} : (Y' \times D) \rightarrow (Y' \times D) \) is the diffeomorphism,
- \( f|_{(Y')} : (Y') \rightarrow \partial Y \) is the simple homotopic equivalence,
- \( f|_{(X \setminus Y')} : (X \setminus E) \rightarrow (X \setminus Y) \) is the simple homotopic equivalence.

On this set we introduce the operation of gluing together two elements according to \( X' \times E' \), which is possible in virtue of the first condition, and since \( D \cup E = D \), then after a smoothing of the angles we obtain an element of the same set. We shall assume that \( (M, f) \sim (X, \partial X) \) if there exists a homotopy \( F : (M \times I, \partial M \times I) \rightarrow (X, \partial X) \) of the mapping \( f \) such that \( F|_{f(Y') \times \partial D} : f(Y') \rightarrow \partial Y \) is a simple homotopic equivalence, and \( F|_{f(Y') \times \partial D} = f|_{f(Y') \times \partial D} \), \( (p : M \times I \rightarrow M \) is the projection), and as a result of the homotopy \( f|_{f(Y') \times \partial D} : f(Y') \rightarrow \partial Y \) is a simple homotopic equivalence. In this case it is said that \( f' \) is split along \( Y \). Let \( T : D \rightarrow D \) be a diffeomorphism of the disk on itself, which changes the orientation. The resulting of gluing together \( (M, f) \) and \( (M', f') \) if is split along \( Y \). We identify \((M_1, f_1) \) and \((M_2, f_2) \) if the result of gluing \((M_1, f_1 \circ T \circ f_1) \) with \((M_2, f_2) \) decomposes. We obtain a group which will be denoted by \( BL(X, Y) \).

1.2. Notation. \( V = X \times I; L(X) = L_{\dim X}(\pi_1(X), \omega (X)) \).

**THEOREM 1.** The following diagram exists:

\[
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta} & BL(X \times I, Y \times I) & \xrightarrow{\iota} & L(Y \times I) & \xrightarrow{r} & L(V) & \xrightarrow{i} & L(X) \\
\end{array}
\]

\( (D) \)

\[
\begin{array}{ccccccccc}
\cdots & \xrightarrow{r} & L(V \times I) & \xrightarrow{i} & L(X \times I) & \xrightarrow{\delta} & BL(X, Y) & \xrightarrow{\iota} & L(Y) \\
\end{array}
\]


© 1975 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.
a) Both rows, which extend to the left without limit, are chain complexes, i.e., the composition of the two homomorphisms is equal to zero.

b) The homologies of the two complexes are isomorphic.

Proof. Diagram D is part of the following diagram, cited the book [1]:

\[
\begin{array}{ccc}
\sim L(V \times I) & \sim L(X \times I) & BL(X, Y) \\
\downarrow f & \downarrow g & \downarrow \beta \\
BL(X \times I, Y \times I) & L(Y \times I) & L(V) \\
\end{array}
\]

Here \( G \) and \( H \) are certain groups. A simple diagram search shows that the upper and lower rows are complexes and their homologies are isomorphic.

One can construct the mappings in diagram D independently of the Wall diagram and prove Theorem 1. The remainder of this article is devoted to the construction of the mappings.

The Mapping \( r : L(Y \times I) \rightarrow L(V) \). We realize the element \( x \) by the normal mapping \( f : N \rightarrow Y \times I \) on the lower boundary of which lies the identity mapping \( Y \rightarrow Y \), and on the upper boundary \( \approx \) a simple homotopic equivalence of a certain manifold in \( Y \). By virtue of the lemma on the extension of a normal cobordism (see, [2], Chap. IV, Sec. 3.3) \( f \) can be extended to the mapping \( F : M \rightarrow X \times I \), on the lower boundary of which lies the identity mapping \( X \rightarrow X \). By construction \( F^{-1}(Y \times I) \rightarrow (Y \times I) \) is a simple homotopic equivalence; therefore the mapping \( F^{-1}|_{(Y \times I)} \rightarrow (X \times I) |_{(X \times I)} \) on the boundary is a simple homotopic equivalence. It determines the element from \( L(V) \), which is set equal to \( r(x) \). If we realize \( x \) by two different methods in the form of a normal mapping \( f : N \rightarrow Y \times I \), then their union is rearranged up to a simple homotopic equivalence. Having here again utilized the lemma on the extension of a normal cobordism, one can easily construct a cobordism between the two normal mappings in \( V \) obtained above, which proves correctness.

For convenience in the description of the mapping \( \beta \), we introduce the following definition. The set of simple homotopic equivalences \( f : M \rightarrow X \) such that \( \partial M \rightarrow \beta M \rightarrow \beta X \) is a diffeomorphism, and where the mappings \( f_1 : M \rightarrow X \) and \( f_2 : M \rightarrow X \) are identified if a diffeomorphism \( h : M_1 \rightarrow M_2 \) exists such that \( f_1 \sim f_2 \) and the homotopy is fixed on the boundary, is called the set of homotopic smoothings of the manifold \( X \) (and is denoted \( hS(X) \)).

Let \( \eta : L(X \times I) \rightarrow hS(X) \) be a left mapping into the exact sequence of Sullivan, and let \( \xi : hS(X) \rightarrow BL(X, Y) \) be the unique mapping of "taking an obstruction to decomposition." The composition of these two mappings \( L(X \times I) \rightarrow hS(X) \rightarrow BL(X, Y) \) is, by definition, the mapping \( \beta \).

The mapping \( c : BL(X, Y) \rightarrow L(Y) \) corresponds to a transition from "obstruction to internal surgery" to "obstruction to abstract surgery," i.e., if \((M, f)\) represents the element \( x \in BL(X, Y) \) then the mapping \( F|_{L(Y)} \rightarrow (Y \times I) \) is normal and gives the element \( c(x) \in L(Y) \).

The mapping \( i : L(V) \rightarrow L(X) \) is induced by the inclusion \( V \subset X \) or, if convenient, is induced algebraically by the mapping \( \pi_1(V) \rightarrow \pi_1(X) \).

Standard geometrical arguments show that the compositions \( \beta \circ i, i \circ r, r \circ c, \) and \( \beta \circ c \) are equal to zero.

Now let us turn to mappings into homologies. An isomorphism into homologies in the term \( L(X \times I) \) and in the term \( L(Y \times I) \) is induced by the mapping \( \alpha : Ker \beta \rightarrow L(Y \times I) / \text{Im} \beta \), which will now be constructed. Let \( x \in Ker \beta \subset L(X \times I) \). We realize \( x \) by the normal mapping \( F : N \rightarrow (X \times I) \). By convention, \( F \) can be deformed into a mapping \( F_1 : N \rightarrow (X \times I) \) such that the mapping \( F_1|_{F_1^{-1}(Y \times I)} : F_1^{-1}(Y \times I) \rightarrow Y \times I \) is a simple homotopic equivalence, but then the mapping

\[ F_1|_{F_1^{-1}(Y \times I)} : F_1^{-1}(Y \times I) \rightarrow (Y \times I) \]

gives the element \( \alpha(x) \in L(Y \times I) \). The ambiguity, equal to \( \text{Im} c \), originates from the arbitrariness in the choice of the mapping \( F_1 \).

The isomorphism into homologies in the term \( L(V) \) and in the term \( BL(X, Y) \) is induced by the mapping \( h : Ker i \rightarrow BL(X, Y) / \text{Im} \beta \). Let \( f : N \rightarrow V \) be a normal mapping, realizing the element \( x \in Ker i \subset L(V) \). We have \( V = X \setminus U, \partial V = \partial X_{or} \cup \partial U_{ov} \). One can assume that the mapping \( f \) is such that \( f|_{U_{ov} \cup \partial U_{ov}} \rightarrow (U_{ov} \cup \partial U_{ov}) \) is a diffeomorphism. Therefore, one can paste \( U \) to the manifolds \( N \) and \( V \) and, having