AN APPROXIMATE METHOD OF SOLVING PROBLEMS OF THE THEORY OF ELASTICITY FOR AN INCOMPRESSIBLE MATERIAL IN THE CASE OF LARGE DEFORMATIONS

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A method is proposed for solving problems with large deformations on the basis of the results of the linear theory of elasticity. The calculations are illustrated by examples, and the results obtained are compared with existing data.

There have been only a few attempts at the approximate calculation of rubber parts operating at large deformations [1–7]. In [1–6] the statistical theory of high elasticity [8] was employed; the form of the elastic potential and its refinement for the construction of the total potential energy of the system were selected as a function of the operating characteristics of each type of part. According to published data, the results of the calculations agree closely with existing experimental data. The practical application of the formulas obtained in [1–6] is impeded by their clumsiness and the need for a graphic solution of a number of intermediate equations.

This paper presents an approximate method for calculating rubber shock absorbers, valves, seals, and other rubber engineering components in the presence of large deformations. With this method, the apparatus of the linear theory of elasticity [9] can be used.

If for parts made of incompressible material the continuous process of deformation is treated as discrete, it is possible to obtain a solution for each individual stage of deformation by the methods of the linear theory of elasticity. The linear solutions for each individual stage are constructed with allowance for the change in the shape of the part as a result of the preceding stages. The existing experimental data [7, 8] show that at strains ε ≈ 50%, the shear modulus G for rubber may be assumed constant. Then summation of the linear solutions makes possible an approximate piecewise-linear solution of the problem for large deformations. From the piecewise-linear solution, by passing to the limits (increasing the number of stages leads to a decrease in the degree of deformation in each individual stage), it is possible to obtain the continuous solution.

**Example 1. Uniaxial compression.** When a cube of height 2h (Fig. 1) is compressed in the z direction by an amount Δ with free expansion (no friction) in the x and y directions, the displacements of the points will be proportional to the coordinates, and they are determined by the exact solution of the linear differential equation [9]

\[ \begin{cases} u_i = A_i x_i ; \\ v_i = A_i y_i ; \\ w_i = -2A_i z_i ; \\ A_i = \frac{\delta_i}{2H_i} ; \\ (i = 1, 2, \ldots, N), \end{cases} \]

where \( x_i, y_i, \) and \( z_i \) are rectangular Cartesian coordinate systems determining the position of points on the cube in each \( i \)-th stage of compression; \( u_i, v_i, \) and \( w_i \) are the displacements of the points in the \( i \)-th stage in the directions \( x_i, y_i, \) and \( z_i, \) respectively; \( \delta_i \) is the compression of the cube in the \( i \)-th stage; \( H_i \) is the height of the cube in the \( i \)-th stage; and \( N \) is the number of stages selected.

Assuming that all the stages are equal, \( \delta_1 = \delta_2 = \ldots = \delta_i = \ldots = \delta, \) we have \( \Delta = \Sigma \delta_i = N \delta. \)

Then from (1),

\[ \begin{cases} A_i = \frac{\delta}{2H_i} ; \\ H_i = h \left[ 1 - (i - 1) \frac{\delta}{2h} \right] ; \\ N \frac{\delta}{2h} = \frac{\Delta}{2h} . \end{cases} \]
From the condition $\sigma_{x1} = \sigma_{y1} = 0$, we find the hydrostatic pressure function $S_i$ in each stage of compression: $S_i = -2A_i$. Then the normal stress $\sigma_{z1}$ is given by

$$\sigma_{z1} = -6GA_i.$$  

From the equilibrium condition,

$$\int_a^r \int \sigma_{z1} \, dx \, dy = -P_i.$$  

Here, $\Omega_i$ is the compression surface area of the cube in the $i$-th stage determined in accordance with (1), (2); $P_i$ is the force needed to compress the cube in each individual $i$-th stage by an amount $\delta_i$. Taking into account the change in the shape of the cube during compression by stages, from (4) we find

$$P = 3FG \left( \frac{1}{\lambda_1} - 1 \right),$$  

where $\lambda = 1 - \Delta/2h$ is the degree of compression, and $F = 4h^2$ is the compression surface area of the cube in the predeformation stage.

The stresses $\sigma_{z1}$ for each $i$-th stage are determined from (3). Then for large strains

$$\sigma_{z1} = \lim_{\delta \to 0} \sum_{i=1}^{N} \sigma_{n_i} = 3G \ln \lambda.$$  

The displacement of the faces of the cube in the $x_i$ and $y_i$ directions in each $i$-th stage is given in accordance with (1), (2) by

$$u_i = hA_i; \quad u_2 = hA_2(1 + A_i); \quad u_3 = hA_3(1 + A_i)(1 + A_2); \quad \ldots$$  

Then for large strains the displacement of the faces of a cube compressed by an amount $\Delta$ is

$$u(h) = v(h) = \lim_{\delta \to 0} \sum_{i=1}^{N} u_i(H_i).$$  

Example 2. Simple tension. This condition is formally identical with uniaxial compression: Eqs. (7), (8), and (10), in which the values of $\lambda$ are now greater than 1 and the quantity $P$ is negative, remain valid.

Thus, in accordance with (7), tension and compression are represented by the same curve.

We will compare our data with the results of the statistical theory of high elasticity. In Fig. 2, in which $f = P/FG$ has been plotted along the ordinate axis, the complete tension-compression diagram is shown; curve 1 was calculated from (7), curve 2 from the statistical theory of high elasticity; the circles represent the experimental data of [8] for the types of deformation considered.

Equation (10) coincides with the analogous relation obtained from the statistical theory of high elasticity.

Example 3. Calculation of the characteristics of a long shock absorber (Fig. 3) compressed by an amount $\Delta$ by a linear load $P_0$ (plane problem). Below, the Ritz method is used to construct the linear solutions in each $i$-th stage of compression; the hypothesis of plane sections is assumed to apply to all the stages.

Assuming that there is no slip between the rubber and the metal, we write the expressions for the dis-