PROBLEMS CONNECTED WITH OPTIMUM QUADRATURE FORMULAS FOR THE FUNCTION CLASSES $H^p_\omega$ AND $H^q_\omega$ ARE INVESTIGATED.

WE CONSIDER APPROXIMATE FORMULAS FOR INTEGRALS

$$\int_a^b \rho(x)f(x)dx,$$

where $f(x)$ is any function of some class $H$ and $\rho(x)$ is a given weight function such that the product $f(x)\rho(x)$ is integrable on $[a, b]$ for any function of $H$. Let

$$\int_a^b \rho(x)f(x)dx = \sum_{k=0}^{n} A_k f(\gamma_k) + E_n[f;\{A_k, \gamma_k\}], a \leq \gamma_0 < \gamma_1 < \ldots < \gamma_n \leq b$$

so that the sum on the right-hand side is an approximation for the integral (1). The error of this approximation for a given function $f(x)$ of $H$ is

$$E_n[f;\{A_k, \gamma_k\}] = \sup_{x \in H} |\int_a^b \rho(x)f(x)dx - \sum_{k=0}^{n} A_k f(\gamma_k)|.$$ 

If the abscissas $a \leq \gamma_0 < \ldots < \gamma_n \leq b$ and the coefficients $A_k$, $k = 0, 1, \ldots, n$, are fixed, the best estimate of the error of the quadrature formula with fixed weight $\rho(x)$ for the class $H$ is

$$E_n[f;\{A_k, \gamma_k\}] = \sup_{x \in H} |\int_a^b \rho(x)f(x)dx - \sum_{k=0}^{n} A_k f(\gamma_k)|.$$ 

If the weight function $\rho(x)$ is given, the least upper bound (4) for a given class of functions depends only on the choice of the abscissas $\gamma_k$ and the coefficients $A_k$. In quadrature theory the problem arises of constructing quadrature formulas of the form (2) with the smallest possible error in a given class of functions: (a) for fixed abscissas, (b) for fixed abscissas and coefficients, (c) for fixed coefficients.

In these three cases the best possible bounds for errors in the case of a weight function $\rho(x)$ in the class $H$ are

$$E_n[H;\{A_k, \gamma_k\}] = \inf_{(A_k)} E_n[H;\{A_k, \gamma_k\}],$$

$$E_n[H;\{A_k, \gamma_k\}] = \inf_{(A_k, \gamma_k)} E_n[H;\{A_k, \gamma_k\}],$$

$$E_n[H;\{A_k, \gamma_k\}] = \inf_{(\gamma_k)} E_n[H;\{A_k, \gamma_k\}].$$

In the following the sum on the right-hand side of (2) will be called the best quadrature formula (a) for the class $H$ with weight $\rho(x)$ and fixed abscissas, (b) for the class $H$ with weight $\rho(x)$, and (c) for the class $H$ with weight $\rho(x)$ and fixed coefficients respectively if (a), (b), or (c) is satisfied.

In the present note we investigate the construction of quadrature formulas of types (a) and (b) for the function classes $H_\omega$ and $H_\omega^{(n)}$ with an arbitrary weight $\omega(x)$. We assume that the interval $[a, b]$ is transformed into $[0, 1]$; it is known that this does not lead to any loss of generality.

We write $H_\omega$ for the class of functions $f(x)$ defined in $[0, 1]$ and satisfying the condition $|f(x_1) - f(x_2)| \leq \omega(x_2 - x_1)$, for any $x_1$ and $x_2$ $(0 \leq x_1 < x_2 \leq 1)$ where $\omega(x)$ is a given continuity modulus.

The definition of $H_\omega^{(n)}$ depends on the abscissas. We say that $f(x) \in H_\omega^{(n)}$ for a given set of abscissas $\{\gamma_k\}, 0 \leq \gamma_0 < \ldots < \gamma_n \leq 1$, if it is defined in $[0, 1]$ and satisfies the conditions

$$|f(x) - f(\gamma_k)| \leq \omega(|x - \gamma_k|), x_k \leq x \leq x_{k+1}, k = 0, 1, \ldots, n,$$

where $x_0 = 0, x_k = (\gamma_{k-1} + \gamma_k)/2, k = 1, 2, \ldots, n-1, n, x_{n+1} = 1$. Plainly $H_\omega \subset H_\omega^{(n)}$ for any set of abscissas and any $n$.

**Theorem 1.** Let $\{\gamma_k\}, k = 0, 1, \ldots, n$, be an arbitrary system of abscissas in the interval $[0, 1]$ and let the coefficients of quadrature formula defined by (2) be

$$A_k = \int_{x_k}^{x_{k+1}} \omega(x) dx,$$

where

$$x_0 = 0, \quad x_k = (\gamma_{k-1} + \gamma_k)/2, \quad k = 1, 2, \ldots, n, \quad x_{n+1} = 1.$$

Then

$$E_n [H_\omega; \{A_k, \gamma_k\}] = E_n [H_\omega^{(n)}; \{A_k, \gamma_k\}] = \frac{1}{n} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \omega(x) dx.$$

In fact if $f(x) \in H_\omega$ or $f(x) \in H_\omega^{(n)}$, we have

$$|E_n[f; \{A_k, \gamma_k\}]| = \left| \frac{1}{n} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \omega(x) dx \right| \leq \frac{1}{n} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \omega(x) dx.$$

It remains to prove that there is a function $\varphi_\omega(x)$ of $H_\omega$ for which we have equality in (6). Let $\varphi_\omega(x) = \omega(x)$ and $\varphi_\omega^{(n)}(x) = \omega(1 - x^n)$. Clearly $\varphi_\omega(x)$ is continuous in $[0, 1]$ since it is continuous in each subinterval $[x_k, x_{k+1}], k = 0, 1, \ldots, n$, and $\varphi_\omega^{(n)}(0) - \varphi_\omega^{(n)}(1) - \omega(1 - x^n) = \omega(1 - x^n)$. Let $x' = 0, x'' = 1$, then

$$|\varphi_\omega(x') - \varphi_\omega(x'')| = |\varphi_\omega(x') - \varphi_\omega(x'')| \leq \omega(x' - x'').$$

If $x', x'' \in [x_k, x_{k+1}], k = 0, 1, \ldots, n$, then

$$|\varphi_\omega(x') - \varphi_\omega(x'')| = |\varphi_\omega(x') - \varphi_\omega(x'')| \leq \omega(x' - x'').$$

Since $\varphi_\omega(x)$ is symmetric with respect to $x = x_{k+1}$, $k = 0, 1, \ldots, n-1$ in every interval $[\gamma_k, \gamma_{k+1}]$, condition (7) is satisfied in these intervals.

We now assume that $x_{k+1} \leq x' \leq x_k + 1, \gamma_m \leq x' \leq x_{m+1}, 0 \leq k < m \leq n-1$, and that $x'$ coincides either with $x'$ or with the point located symmetrically with respect to $x_{k+1}$, and $x''$ either with $x''$ or with the point symmetrically located with respect to the point $x_{m+1}$; and the symmetry of $\varphi_\omega(x)$ in the intervals $[\gamma_m, \gamma_{m+1}]$ with respect to $x = x_{m+1}$ implies that

$$\varphi_\omega(x') - \varphi_\omega(x'') = |\varphi_\omega(x') - \varphi_\omega(x'')| = |\omega(x_{k+1} - x')$$

$$- \omega(x' - \gamma_m)| \leq \max \{\omega(x_{k+1} - x'), \omega(x' - \gamma_m)| \leq \omega(x' - x'').$$

If $\gamma_k \leq x' \leq \gamma_{k+1}, \gamma_m \leq x' \leq x_{m+1}, 0 \leq k < m \leq n-1$. Let $0 \leq x' \leq \gamma_m, \gamma_k \leq x' \leq x_{k+1}, k = 0, 1, \ldots, n$, let $x'$ coincide with either $x'$ or with the point located symmetrically with respect to $x_{k+1}, k = 0, 1, \ldots, n-1$, and let $x' = x'$, if $\gamma_n \leq \gamma_m \leq 1$. Then

$$\varphi_\omega(x') - \varphi_\omega(x'') = |\varphi_\omega(x') - \varphi_\omega(x'')| = |\omega(x' - \gamma_n)$$

$$- \omega(x' - x'')| \leq \max \{\omega(x' - \gamma_n), \omega(\gamma_n - x'')\} \leq \omega(x' - x')$$.

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