\[
\lim \inf_{\varepsilon \to 0} \sup_{T(x) \in \mathcal{S}} \mathbb{E}_S(\|T(x) - \| S_0 \|_{p, U} \|/(\varepsilon^2 \cdot \ln(1/\varepsilon))^{(\beta-\beta)/(2\beta+1-\beta/\beta)}) > 0.
\]

Let us now observe that the function \( h \) in (21) can be chosen such that
\[
\max h^{(j)}(t) = \max | h^{(j)}(M) |.
\]

Therefore, the supremum in the last inequality may be considered over only those \( S \) for which
\[
\max S^{(j)}(t) = \| S^{(j)} \|_{\infty, U}.
\]

Hence, setting \( p = \infty \), we get (20). The theorem is proved.

**LITERATURE CITED**


**LIMIT THEOREMS FOR A CRITICAL BRANCHING PROCESS WITH IMMIGRATION**

M. Kh. Asadullin and S. V. Nagaev

1. **Discrete Time.** We consider the following model of a branching process with immigration. Let \( \xi_0, \xi_1, \xi_2, \ldots \) be nonnegative integer-valued random variables. The random variable \( \xi_1 \) will be interpreted as the number of particles immigrating into some population from the outside at time 1. Each of the particles immigrating at time \( i \) gives rise to a branching process \( Y_i^{(j)}(t), 1 \leq j \leq \xi_i \). We assume that the processes \( Y_i^{(j)}(t) \) are mutually independent and for any \( i, j \) the process \( Y_i^{(j)}(t) \) has the same distribution as \( Y_0^{(j)}(t-i) \), where \( Y_0^{(j)}(t) \) is the branching process generated by one particle at time 0.

Let \( Z(n) \) be the total number of particles in the population at time \( n \). If there are no particles in the population at the beginning, then
\[
Z(n) = \sum_{i=0}^{n} \sum_{j=1}^{\xi_i} Y_i^{(j)}(n).
\]

Our goal is to study the asymptotic behavior of the distribution of \( Z(n) \) as \( n \to \infty \).

The simplest case is when \( Y_0^{(j)}(n) \) is a Galton–Watson process and the immigration process is given by a sequence of independent identically distributed random variables. Results concerning the asymptotic behavior of \( Z(n) \) in this case may be found, for example, in [1, Chap. VI, Sec. 7].

In [2] it is assumed that \( \xi_0, \xi_1, \xi_2, \ldots \) form a stationary sequence in the wide sense, satisfying the condition \( \text{Cov}(\xi_0, \xi_n) \to 0 \) as \( n \to \infty \), and \( Y_i^{(j)}(t) \) belongs to a certain class containing those critical Bellman–Harris processes for which the lifetime of particles and the number of particles generated by one particle have finite second moments.
In this paper we consider a more general situation from the viewpoint of restrictions on the incoming process.

We shall say that a sequence \( \{\xi_i\}_{i=0}^{\infty} \) of random variables satisfies condition (A) if there exists a random variable \( \xi \) such that 
\[
\lim_{n \to \infty} n^{1-s} \mathbb{E} \left| \sum_{i=0}^{n} (\xi_i - \xi) \right| = 0.
\]

The following result will play an important role in what follows.

**Lemma 1.** If a sequence \( \{\xi_i\}_{i=0}^{\infty} \) satisfies condition (A), then
\[
\sum_{i=0}^{n} \frac{\xi_i}{n - i + n s^{-1} - \delta} \rightarrow_{n \to \infty} n \xi \ln (1 + s),
\]
where \( s > 0 \).

**Proof.** It is easy to see that
\[
\sum_{i=0}^{n} \frac{\xi_i}{n - i + n s^{-1} - \delta} = \frac{1}{n - i + n s^{-1} - \delta} \cdot \sum_{i=0}^{n} \frac{1}{n - i + n s^{-1}} = n^{-1} \sum_{i=0}^{n} S_i (c_i - c_{i+1}) + s n^{-1} S_n,
\]
where \( c_i = \frac{n}{n - i + n s^{-1}} \), \( S_i = \sum_{j=0}^{i} (\xi_j - \xi) \).

We note that
\[
|c_i - c_{i+1}| \leq s^2 n^{-1}
\]
uniformly in \( i, 0 \leq i \leq n - 1 \).

From this we obtain
\[
\mathbb{E} \left| n^{-1} \sum_{i=0}^{n-1} S_i (c_i - c_{i+1}) + s n^{-1} S_n \right| \leq \left( \text{max}_{0 \leq i \leq n-1} |c_i - c_{i+1}| \right) \sum_{i=0}^{n-1} \mathbb{E} |S_i| + s n^{-1} \mathbb{E} |S_n| \leq s^2 n^{-2} \sum_{i=0}^{n-1} \mathbb{E} |S_i| + s n^{-2} \mathbb{E} |S_n|.
\]

Since
\[
n^{-2} \sum_{i=0}^{n-1} \mathbb{E} |S_i| \leq n^{-1} \sum_{i=1}^{n-1} \mathbb{E} |S_i| + n^{-2} \mathbb{E} |S_0|
\]
and by condition (A),
\[
i^{-1} \mathbb{E} |S_i| \rightarrow_{i \to \infty} 0,
\]
we have
\[
n^{-2} \sum_{i=0}^{n-1} \mathbb{E} |S_i| \rightarrow_{n \to \infty} 0.
\]

By inequality (3) this means that
\[
\mathbb{E} \left| n^{-1} \sum_{i=0}^{n-1} S_i (c_i - c_{i+1}) + s n^{-1} S_n \right| \rightarrow 0.
\]

Returning to the representation (2), we obtain
\[
\sum_{i=0}^{n} \frac{\xi_i}{n - i + n s^{-1} - \delta} \rightarrow_{n \to \infty} n \xi \ln (1 + s).
\]
On the other hand,
\[
\sum_{i=0}^{n} \frac{1}{n - i + n s^{-1}} = \frac{1}{n} \sum_{i=0}^{n} \frac{1}{(n-i) n s^{-1}} \rightarrow_{n \to \infty} \ln (1 + s).
\]
Consequently,
\[
\sum_{i=0}^{n} \frac{\xi_i}{n - i + n s^{-1} - \delta} \rightarrow_{n \to \infty} n \xi \ln (1 + s),
\]
which was to be proved.