ON A CLASSIFICATION OF MEROMORPHIC
FUNCTIONS DEFINED IN THE FINITE PLANE

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We consider a subclass of meromorphic functions in the finite plane \(|z| < \infty\), which generalizes Yosida's subclass of functions of the first category. We prove necessary and sufficient conditions for a function to belong to this subclass.

1. In [1] K. F. Tse divided the set of all meromorphic functions defined in the unit disk \(|z| < 1\) into two nonintersecting subsets: meromorphic functions of the first type and of the second type. He proved several necessary and sufficient conditions for a function to belong to the first type and he also studied the distribution of values and boundary properties of meromorphic functions of the first type. Both sets are nonempty: meromorphic functions of the first type, as defined by Tse, are normal meromorphic functions of the first category, introduced and studied by K. Nosiro [2].

K. Nosiro's investigations [2] were prompted by K. Yosida's studies [3] of class (A) meromorphic functions \(f(z)\), defined in the finite plane \(|z| < \infty\) and possessing the property that for an arbitrary infinite sequence of complex numbers \(\{a_n\}\), \(|a_n| \to \infty\), the family

\[ \{f_n(z)\}, f_n(z) = f(z + a_n) \quad (n = 1, 2, \ldots) \]

is normal in the sense of Montel in \(|z| < \infty\). If, in addition, no family \(\{f_n(z)\}\) contains identically constant functions as limiting functions, then such a class (A) function was said by K. Yosida to be a function of the first category. (For other properties of class (A) functions, see [4], [5].)

In the present paper, developing Yosida's and Tse's ideas, we divide all meromorphic functions, defined in the finite plane \(|z| < \infty\), into two nonintersecting subsets: meromorphic functions of the first type and of the second type. We prove necessary and sufficient conditions for a function to belong to the first type and we study the distribution of values and the asymptotic properties of functions of the first type. Our theorems are analogous to K. F. Tse's results [1] for the unit disk and their proofs are partially based on Tse's ideas.

2. Let \(w = f(z)\) be a meromorphic function defined in the finite plane \(|z| < \infty\) and assuming values in the extended complex plane \(\Omega\). We call the quantity

\[ \rho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^p} \]

the spherical derivative of the function \(w = f(z)\).

We introduce the following definition.

Definition 1. The function \(w = f(z)\), meromorphic in the finite plane \(|z| < \infty\), belongs to the second type if there exists a sequence of points \(\{a_n\}\), \(|a_n| \to \infty\), such that the family of functions

\[ \{f_n(z)\} = \{f(z + a_n)\} \]

converges uniformly to a constant in some neighborhood of the point \(z = 0\).
A meromorphic function belongs to the first type if it does not belong to the second type.

It follows immediately from Definition 1 that an arbitrary meromorphic function of the first category in \( K \), Yosida's classification [3] is a function of the first type in the sense of Definition 1.

Definition 1 was introduced by V. I. Gavrilov, who also proved the following theorem.

**Theorem 1** (V. I. Gavrilov). The function \( w = f(z) \), meromorphic in the finite plane \( |z| < \infty \), belongs to the first type if and only if for each \( r > 0 \) there exists a \( \delta = \delta(r) > 0 \) such that

\[
\liminf_{|z| \to \infty} \left( \sum_{|z| < r} |p(f(z))|^2 \right) \, dx \, dy = \delta, \quad z = x + iy. \tag{1}
\]

**Proof. Sufficiency.** Let us assume, on the contrary, that the function \( w = f(z) \) satisfies the condition (1) and belongs to the second type. Then there exists a sequence of points \( (a_n), |a_n| \to \infty \), such that the sequence of functions \( \{f_n(z)\} = \{f(z + a_n)\} \) converges uniformly to a function identically constant in some disk \( |z| < r, \quad 0 < r < \infty \). Consequently, \( \lim_{n \to \infty} \rho(f_n(z)) = 0 \) uniformly in \( |z| < r \), and

\[
\lim_{n \to \infty} \sum_{|z| < r} |p(f(z))|^2 \, dx \, dy = \lim_{n \to \infty} \sum_{|z| < r} |p(f_n(z))|^2 \, dx \, dy = 0, \tag{2}
\]

Thus we have arrived at a contradiction with the condition (1).

**Necessity.** Assume that for the function \( w = f(z) \) of the first type we can produce a number \( r > 0 \) and a sequence of points \( (a_n), \quad |a_n| \to \infty \), such that the sequence of functions \( \{f_n(z)\} = \{f(z + a_n)\} \) possesses the property (2).

We show, on the one hand, that \( (a_n) \) must be a sequence of \( M(z) \)-points for the function \( w = f(z) \) and, on the other hand, that it cannot be such a sequence, thereby arriving at a contradiction. (For properties of sequences of \( M(p) \)-points, \( p \geq 1 \), see [6, 7].)

Actually, if \( (a_n) \) is not a sequence of \( M(z) \)-points, then by definition there exists a number \( r > 0 \) and a subsequence \( \{a_m\} \) of the sequence \( \{a_n\} \) such that the function \( w = f(z) \) does not assume three distinct values on the set \( \bigcup_{m} \{ |z - a_m| < r \} \). In other words the family of functions \( \{f_m(z)\} = \{f(z + a_m)\} \) \( (m = 1, 2, \ldots) \) is normal in the disk \( |z| < r \), and from the family we can select a subsequence \( \{f_{m_k}(z)\} \) which converges uniformly in the disk \( |z| < r \) to a meromorphic function \( g(z) \). Since \( w = f(z) \) is a function of the first type, we have \( g(z) \neq \text{const} \). By virtue of the uniform convergence in the disk \( |z| < r \), we obtain \( \lim_{k \to \infty} \rho(f_{m_k}(z)) = \rho(g(z)) \), whence

\[
\lim_{k \to \infty} \sum_{|z| < r} |p(f_{m_k}(z))|^2 \, dx \, dy = \lim_{k \to \infty} \sum_{|z| < r} |p(g(z))|^2 \, dx \, dy > 0, \quad z = x + iy.
\]

This contradicts the condition (2). Thus the sequence \( (a_n) \) must be a sequence of \( M(z) \)-points of the function \( w = f(z) \).

On the other hand, an arbitrary sequence of \( M(z) \)-points of the meromorphic function \( w = f(z) \) contains a subsequence \( \{a_m\} \) which is a sequence of \( \mu(z) \)-points of the function \( w = f(z) \), i.e., a subsequence \( (a_m), \quad |a_m| \to \infty \), such that for each \( m \) the function \( w = f(z) \) assumes all values from \( \Omega \) in the disk \( |z - a_m| < 1/m \), except possibly the values from two sets \( E_m \) and \( G_m \), whose diameters are less than \( 2/m \). (For the definition and properties of sequences of \( \mu(p) \)-points, \( p \geq 1 \), see [7].)

Therefore, if \( (a_n) \) were a sequence of \( M(z) \)-points of the function \( w = f(z) \), then the integral

\[
\sum_{|z| < r} |p(f_m(z))|^2 \, dx \, dy = \sum_{|z| < r} |p(f(z))|^2 \, dx \, dy,
\]

expressing the area of the image of the disk \( |z - a_m| < r \) under the mapping \( w = f(z) \), would not be less than \( \pi - \varepsilon_n \), where \( \varepsilon_n > 0 \), \( \lim_{n \to \infty} \varepsilon_n = 0 \). And this would again contradict the condition (2). Thus the theorem is proved in full.

3. We establish yet another criterion for a meromorphic function to belong to the first type.

**Theorem 2.** The function \( w = f(z) \), meromorphic in the finite plane \( |z| < \infty \), belongs to the first type if and only if for arbitrary \( r > 0 \)

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