THE V. A. STEKLOV PROBLEM IN THE THEORY
OF ORTHOGONAL POLYNOMIALS

B. L. Golinskii

An exact order of growth of Szego polynomial kernels and moduli of polynomials orthogonal
in the unit circle is obtained in the zeros of a weight function of special form. Other ques-
tions are also examined.

1. The orthonormal polynomials \( \{\varphi_n(e^{i\theta})\}_{n=0}^{\infty} \) in the unit circle \( z = e^{i\theta}, -\pi \leq \theta \leq \pi \) satisfy the relationships

\[
\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(e^{i\theta}) \overline{\varphi_m(e^{i\theta})} d\sigma(\theta) = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots, \tag{1}
\]

relative to the imposition \( d\sigma(\theta) \), where \( \sigma(\theta) \) is a distribution function; a nondecreasing bounded function with a nondenumerable set of growth points. The function \( \sigma'(\theta) = \varphi(\theta) \) existing almost everywhere is called a weight function if \( \sigma(\theta) \) is absolutely continuous on the segment \([-\pi, \pi]\), and is denoted thus: \( \sigma(\theta) \in AC(-\pi, \pi) \). If \( \sigma(\theta) \) is absolutely continuous only in part of the segment \([\alpha, \beta] \subseteq (-\pi, \pi)\), then this is the notation: \( \sigma(\theta) \in AC(\alpha, \beta) \). For a given distribution function \( \sigma(\theta) \) the system \( \{\varphi_n(z)\}_{n=0}^{\infty} \) is defined by the known formulas

\[
\varphi_n(z) = \chi_n z^n + \ldots + \chi_0, \quad n = 0, 1, 2, \ldots \tag{2}
\]

If \( \sigma(\theta) \in AC(-\pi, \pi) \), then the polynomials \( \varphi_n(z) \) are denoted by \( \varphi_n(z, \varphi) \). In the broad sense as it is understood in [1], pp. 293-298, say, the V. A. Steklov problem includes all problems where the behavior of a sequence of orthonormal polynomials and Szego polynomial kernels

\[
K_n(\theta) = K_n(z_0, z_0), \quad z_0 = e^{i\theta},
\]

must be determined by means of the properties of the weight function.

2. THEOREM 1. If \( \sigma'(\theta) > 0 \) almost everywhere in \([-\pi, \pi]\) then for any point \( \theta_0 \in [\alpha + 2\varepsilon, \beta - 2\varepsilon] \), \([\alpha, \beta] \subseteq [-\pi, \pi]\), \( \varepsilon > 0 \) we have, starting with some \( n \geq n_0 \):

\[
K_n(\theta_0) \left[ \sigma \left( \theta_0 + \frac{1}{2n} \right) - \sigma \left( \theta_0 - \frac{1}{2n} \right) \right] \leq 8n. \tag{3}
\]

Let us first prove that the inequality

\[
\frac{1}{2\pi} \int_{\theta_0}^{\theta_0 + 2\pi} \left| H_n(e^{i\theta}, e^{i\phi}) \right|^2 d\sigma(\theta) \geq \frac{1}{2\pi} \left[ \sigma \left( \theta_0 + \frac{1}{2n} \right) - \sigma \left( \theta_0 - \frac{1}{2n} \right) \right] \tag{4}
\]

is valid in each of the cases 1) and 2) considered below. Here

\[
H_n(z, z_0) = K_n(z, z_0) K_n^{-1}(z_0, z_0).
\]

For \( \theta \in [\alpha, \beta] \) we have: \( |\theta - \theta_0| \leq \eta, \quad \pi/2 \leq |\theta - \theta_0| \leq \pi, \quad 0 < \eta < \varepsilon \leq \pi/2, \quad \theta_0 \in [\alpha + 2\varepsilon, \beta - 2\varepsilon] \), \([\alpha, \beta] \subseteq [-\pi, \pi]\). Let us use the method of proof by contradiction to prove the inequality (4). Thus, there exists a subsequence \( \{n_k\}_{k=0}^{\infty} \), such that

Let us consider a new polynomial of degree $\nu = \lfloor \frac{v}{n_k} \rfloor$:

$$g_\nu(z, z_0) = H_{n_k}(z, z_0) \left( \frac{\frac{1}{2\pi} \sum_{m=1}^{n_k} \left( \frac{1}{2\pi} \sum_{m=1}^{n_k} \frac{1}{m} \right)^\nu - \nu \frac{1}{n_k} }{1 - \frac{9}{2\pi} \sin^2 (\theta - \theta_0) \left( \frac{1}{2\pi} \sum_{m=1}^{n_k} \frac{1}{m} \right)^\nu} \right), \quad g_\nu(z_0, z_0) = 1.$$  

We have

$$|g_\nu(e^{\theta_0}, e^{\theta_0})|^2 \leq |H_{n_k}(e^{\theta_0}, e^{\theta_0})|^2 \left( 1 - \frac{9}{2\pi} \sin^2 (\theta - \theta_0) \left( \frac{1}{2\pi} \sum_{m=1}^{n_k} \frac{1}{m} \right)^\nu \right) \leq |H_{n_k}(e^{\theta_0}, e^{\theta_0})|^2.$$  

It is known ([3], p. 14) that

$$K_{n_k}^{-1}(\theta_0) = \min \left\{ \frac{1}{2\pi} \sum_{m=1}^{n_k} |G_n(z)|^2 |G_n(z)|^2 \, dz(\theta) \left( \theta = e^{i\theta} \right), \right.$$  

where $\{G_n(z)\}_{n=1}^\infty$ is a set of polynomials of $n$-th degree. Let us replace the polynomial $G_n(z)G_n^{-1}(z_0)$ in (6) by $g_{\nu}(z, z_0)$. Then

$$K_{n_k}^{-1}(\theta_0) \leq \frac{1}{2\pi} \sum_{m=1}^{n_k} |g_\nu(e^{\theta_0}, e^{\theta_0})|^2 \, dz(\theta).$$  

Let us use the notation $\epsilon + \epsilon' = [-\pi, \pi]$. For $\theta \in \epsilon'$ we have $0 < \eta \leq |\theta - \theta_0| < \pi/2$, from which

$$1 - \frac{9}{2\pi} \sin^2 (\theta - \theta_0) < 1 - \lambda_0^2 \eta^2 \quad \left( \lambda_0 = \frac{2}{2\pi} \right).$$  

Therefore

$$\frac{1}{2\pi} \sum_{m=1}^{n_k} |g_\nu(e^{\theta_0}, e^{\theta_0})|^2 \, dz(\theta) \leq \left( 1 - \lambda_0^2 \eta^2 \right)^\frac{1}{2\eta} \nu.$$  

Taking (5) into account we obtain

$$\frac{1}{2\pi} \sum_{m=1}^{n_k} |H_{n_k}(e^{\theta_0}, e^{\theta_0})|^2 \, dz(\theta) > \frac{1}{2\pi} \sum_{m=1}^{n_k} |H_{n_k}(e^{\theta_0}, e^{\theta_0})|^2 \, dz(\theta) > \left( 1 - \lambda_0^2 \eta^2 \right)^\frac{1}{2\eta} \nu.$$  

But

$$\frac{1}{2\pi} \sum_{m=1}^{n_k} |H_{n_k}(e^{\theta_0}, e^{\theta_0})|^2 \, dz(\theta) = K_{n_k}^{-1}(\theta_0).$$  

Thus

$$K_{n_k}^{-1}(\theta_0) > \left( 1 + \lambda_0^2 \eta^2 \right)^\frac{1}{2\eta} \nu \left( K_{n_k}^{-1}(\theta_0) - \frac{1}{2\pi} \left[ \frac{1}{2\pi} \sum_{m=1}^{n_k} \frac{1}{m} \right] \left( \theta_0 - \frac{1}{2\pi} \right) \right).$$  

Let us first examine case 1): $\theta_0$ is a point of discontinuity in the continuity of $\sigma(\theta)$. Then [4]

$$\lim_{n_k \to \infty} 2\pi K_{n_k}^{-1}(\theta_0) = \sigma(\theta_0) - \sigma(\theta_0 - 0) = \mu_0.$$  

Hence an absurd equality is obtained from (7):

$$\lim_{n_k \to \infty} \left( 1 + \lambda_0^2 \eta^2 \right)^\frac{1}{2\eta} \nu \leq \frac{4}{3}.$$  

Now, let us examine case 2): $\sigma'(\theta_0) > 0$. Then

$$(1 + \lambda_0^2 \eta^2)^\frac{1}{2\eta} \nu < K_{n_k}^{-1}(\theta_0) : \left( K_{n_k}^{-1}(\theta_0) - \frac{1}{2\pi} \left[ \frac{1}{2\pi} \sum_{m=1}^{n_k} \frac{1}{m} \right] \left( \theta_0 - \frac{1}{2\pi} \right) \right).$$