BIHARMONIC FUNCTIONS, NONNEGATIVE IN A HALF-STRIP

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It is proved that a function which is biharmonic in a half-strip and satisfies homogeneous Dirichlet conditions at the base of the strip oscillates.

It is established that a nonnegative function which is biharmonic in the strip \( \Pi^+ = \{(x, y), x > 0, y \in (0,1)\} \) and satisfies zero Dirichlet boundary conditions for \( y = 0, y = 1 \), is identically zero. The proof of this is divided into several stages, the principal of which are, in the opinion of the authors, of independent interest and are formulated as lemmas.

1. Formulation of the Problem. Estimate of the Growth of Positive Solutions. Consider the nonnegative solution \( u(x, y) \) of the biharmonic equation

\[
\Delta^2 u = \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} \right) = 0
\]

in the half-strip \( \Pi^+ \), which satisfies the boundary conditions

\[
u_{|y=0} = \frac{\partial u}{\partial y} \bigg|_{y=0} = u \bigg|_{y=1} = \frac{\partial u}{\partial y} \bigg|_{y=1} = 0 \quad (x > 0).
\]

Put

\[
\Pi_x^a = \{(x, \eta), |x-a| < a, \eta \in (0,1)\}; \quad \Pi_x^a = \sum |u(x, \eta)| \, dx \, d\eta.
\]

It follows from the results of [1] (see also [2, 3]) that the following lemma holds:

**Lemma 1.** There can be found a positive constant \( c \) such that for any solution \( u(x, y) \) of Eq. (1) which is nonnegative in \( \Pi^+ \) we have

\[
\|v\|_{L_1} \leq c \|v\|_{L_1} e^c(x > 2).
\]

Let us continue the function \( u(x, y) \) on the whole strip \( \Pi = \{(x, y), x \in (-\infty, \infty), y \in (0,1)\} \); to do this we consider the infinitely differentiable step function \( \theta(x) \), equal to unity for \( x \geq 2 \) and zero for \( x \leq 1 \); then the function \( \nu(x, y) = u(x, y) \theta(x) \) is defined in \( \Pi \), and satisfies the condition

\[
\Delta^2 \nu(x, y) = f(x, y) \quad \text{in} \quad \Pi
\]

and the boundary conditions

\[
\nu \bigg|_{y=0} = \nu \bigg|_{y=1} = 0 \quad \quad (-\infty < x < \infty).
\]

Now \( f(x, y) \) is an infinitely differentiable function [since the problem (1)-(2) is hypoelliptic], \( f(x, y) = 0 \) for \( x \leq 1, x \geq 2 \).

**Lemma 2.** For \( \nu(x, y) \) and all its derivatives we have
\[
\left| \frac{\partial^k v}{\partial x^k \partial y^k} (x, y) \right| \leq C_{b, h} e^{c_k}, \quad (8)
\]

where \( c \) is the constant in (3).

**Proof.** The bound (6) directly follows from the inequality (3) and the integral representation of \( v(x, y) \) which we give below.

Consider the domain \( G_{x_0} \) with smooth boundary \( \partial G_{x_0} \), \( \Pi_{10}^* \subset G_{x_0} \subset B_{10}^* \); let \( \theta_1(x, y) \) denote the step function \( \theta_1(x, y) = 1 \) for \( (x, y) \in \Pi_{10}^* \), \( \theta_1(x, y) = 0 \) for \( (x, y) \notin \Pi_{10}^* \). Let \( E(x, y; \xi, \eta) \) be Green's function for the homogeneous Dirichlet problem for the domain \( G_{x_0} \). We use Green's theorem, which in this case has the form

\[
\int_{\partial G_{x_0}} \left( \omega \frac{\partial v}{\partial n} - \nu \Delta w \right) dS = \int_{\Pi_{10}^*} \left( w \frac{\partial \nu}{\partial n} + \Delta \omega \frac{\partial w}{\partial n} - \Delta w \frac{\partial w}{\partial n} + v \frac{\partial \Delta w}{\partial n} \right) dS. \quad (7)
\]

We take \( G = G_{x_0} \setminus \{(\xi - x)^2 + (\eta - y)^2 < E^2\} \), \( (x, y) \in \Pi_{10}^* \), \( v(\xi, \eta) = 0 \), \( \omega(\xi, \eta) \). \( u(\xi, \eta) = E(\xi, \eta; x, y) \) \( \theta_1 \) and note that the integral along \( \partial G_{x_0} \) in (7) is zero. Indeed, for segments of the boundary lying on the straight lines \( y = 0, y = 1 \), this follows from the fact that each term of the integral contains one of the functions \( v, \partial v/\partial y, E, \partial E/\partial y \), which vanishes on that part of the boundary; the vanishing of the integral on the remaining part of the boundary follows from the vanishing of the step function \( \theta_1(\xi, \eta) \), together with all its derivatives. Then, passing to the limit as \( E \to 0 \), and, using the well-known property of Green's function \( E(\xi, \eta; x, y) \), we arrive at the integral representation

\[
u(x, y) = \int_{\Pi_{10}^*} \Delta_{10} \left[ \theta_1(\xi, \eta) E(\xi, \eta; x, y) \right] u(\xi, \eta) \, d\xi \, d\eta. \quad (8)
\]

It should be noted that \( \Delta_{10} \left[ \theta_1(\xi, \eta) E(\xi, \eta; x, y) \right] = 0 \) for \( (\xi, \eta) \notin \Pi_{10}^* \), \( (\xi, \eta) \in \Pi_{10}^* \), and so (8) can be written as

\[
u(x, y) = \int_{\Pi_{10}^* \setminus \Pi_{10}^*} \Delta_{10} \left[ \theta_1(\xi, \eta) E(\xi, \eta; x, y) \right] u(\xi, \eta) \, d\xi \, d\eta.
\]

If \( (x, y) \in \Pi_{10}^* \), the integrand is a function which is infinitely differentiable with respect to \( x, y \) and we have the following equation for the derivatives of \( u(x, y) \):

\[
\frac{\partial^{k+l} u}{\partial x^k \partial y^l} (x, y) = \int_{\Pi_{10}^* \setminus \Pi_{10}^*} \Delta_{10} \left[ \theta_1(\xi, \eta) \frac{\partial^{k+l}}{\partial x^k \partial y^l} E(\xi, \eta; x, y) \right] u(\xi, \eta) \, d\xi \, d\eta. \quad (9)
\]

From (9) and (3) there directly follows the bound

\[
\left| \frac{\partial^{k+l} u}{\partial x^k \partial y^l} (x, y) \right| \leq C_{b, h} e^{c_kl} \leq C_{b, h} e^{c_{k+l}} = C_{b, h} e^{c_k}.
\]

Lemma 2 is proved.

2. Application of the Fourier Transformation. Asymptotic Representation of the Solution. It follows from Lemma 5 that the problem (4), (5) can be solved using the complex Fourier transformation

\[
V(y, \lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} v(x, y) \, dx, \quad (10)
\]

\[
\lambda = \sigma + i\tau; \tau < -c, \quad (11)
\]

Then we obtain the problem of determining the function \( V(y, \lambda) \) satisfying the equation

\[
\left( \frac{d^2}{dy^2} - \lambda^2 \right)^2 V(y, \lambda) = F(y, \lambda), \quad F(y, \lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x, y) \, dx, \quad (12)
\]

\[
V \big|_{y=0} \frac{dV}{dy} \big|_{y=0} = V \big|_{y=1} - \frac{dV}{dy} \big|_{y=1} = 0. \quad (13)
\]

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