DISCRETE GROUPS GENERATED BY REFLECTIONS
IN THE FACES OF SYMPLICIAL PRISMS
IN LOBACHEVSKIAN SPACES

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The paper provides the Coxeter schemes of all the discrete groups generated by reflections in the faces of simplicial prisms the combinatorial type of which is the product of a simplex and a segment in Lobachevskian spaces.

Among the discrete groups of motions, particular interest inheres in the groups generated by finite numbers of reflections in the hyperplanes of Euclidean space or of Lobachevskian space.

Such a group is unequivocally determined by its fundamental region which may be an arbitrary convex polyhedron with dihedral angles of the form $\pi/k$, where $k$ is an integer. We shall call such a polyhedron cellular (although, in contradistinction to the use of this term in [1], the polyhedron is not here assumed to be regular). The images of a cellular polyhedron under the action of a group generated by reflections in its faces fill the entire space without being laid one upon another. In the Euclidean case, a complete description of cellular polyhedra was obtained back in 1934 by Coxeter [2]. The analogous result for Lobachevskian space still does not exist but certain particular cases have been treated. In 1950, Lanner described all the cellular simplexes [3] and then many examples were constructed by V. S. Makarov [4-6], E. B. Vinberg [7, 8], while E. M. Andreev completely investigated the three-dimensional case [9].

In the present paper we describe all the cellular prisms the combinatorial type of which is the product of a simplex and a segment.

Let $\Lambda^n$ be an $n$-dimensional Lobachevskian space, and in it let there exist prism $P$ of the aforementioned combinatorial type with nonobtuse dihedral angles. Since it has been proved [10] for convex polyhedra without obtuse angles that the dimensionality of the intersection of the hyperplanes containing certain faces equals the dimensionality of the intersection of these faces, it is possible to construct hyperplane $\Pi_0$ orthogonal to all the lateral faces of prism $P$. This hyperplane either contains one of the bases or does not intersect any of them but cuts the prism in two (since all the angles are nonobtuse). Therefore, it suffices to know the prisms in which one of the bases is orthogonal to all the lateral faces. This base $\Gamma_0$ (an $n - 1$-dimensional simplex) and angles $\alpha_1, \ldots, \alpha_n$ formed by the second base $\Gamma_{n+1}$ with the lateral faces $\Gamma_1, \ldots, \Gamma_n$ completely determine the prism. For its existence, obviously, it is necessary that there exist simplex $\Gamma_0$ (which, certainly, is cellular) and all the simplicial angles with vertices $\Gamma_{n+1}$, and that not all the $\alpha_i$ be right angles. Let us prove the sufficiency of these conditions.

We construct $\Gamma_0$ and the hyperplanes $\Pi_i$ ($1 \leq i \leq n$) of the lateral faces orthogonal to it, and we choose the "upper" semi-space (with respect to hyperplane $\Pi_0$) in which our prism must lie; this determines in a natural way the numbering of all the angles. We choose one of the edges orthogonal to $\Gamma_0$ and let it intersect the faces $\Gamma_1, \ldots, \Gamma_{n-1}$; through each of its points $Q$ we can pass a hyperplane $\Pi_Q$ forming, with each $\Pi_i$ (except, perhaps, $\Pi_n$), angle $\alpha_i$. It can be assumed that when $1 \leq i \leq n - 1$ not all the $\alpha_i$ are right angles so that then, if $Q$ is placed at a vertex of the "lower" base $\Gamma_0$, the angle between $\Pi_Q$ and $\Pi_n$ will be obtuse. As $Q$ moves "upward" along the edge (towards infinity), this angle continuously decreases and, from some moment on, $\Pi_Q$ and $\Pi_n$ cease to intersect at all. This means that the angle between $\Pi_Q$ and $\Pi_n$
can assume any specified value \( \alpha_n \) (0 < \( \alpha_n \) < \( \pi/2 \)), and \( \Pi_0 \) with this will still not intersect \( \Pi_0 \), so that all the dihedral angles of the prism turn out to be nonobtuse. But since, by hypothesis, there also exist the other angles at the vertices of the "upper" base, we have indeed constructed the requisite prism.

It can certainly be assumed that some (not all) vertices lie on the absolute, so that we will have obtained an unbounded prism with a finite volume. A vertex on the absolute corresponds to Euclidean simplexes while those in ordinary regions correspond to simplexes on a sphere.

Since all the simplexes with angles \( \pi/k \) in Euclidean space, on a sphere, and in Lobachevskian space are known [2, 3], we can then find all the prisms in \( \Lambda^n \) with such angles. They exist only when \( n \leq 5 \) (since cellular simplexes in \( \Lambda^n \) exist only when \( n \leq 4 \)).

It is convenient to represent our results in the form of Coxeter graphs [to each fact there corresponds a small circle, to angle \( \pi/k \) either a \((k-2)\)-fold line or a simple line with index \( k \), and a dashed line means that the faces do not intersect].

**Bounded Prisms in \( \Lambda^3 \)**

![Diagram of bounded prisms in \( \Lambda^3 \)]

**Unbounded Prisms of Finite Volume in \( \Lambda^3 \)**

![Diagram of unbounded prisms of finite volume in \( \Lambda^3 \)]