A BISINGULAR EQUATION WITH TRANSLATION
IN THE SPACE Lp

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In this paper we use a local method to introduce and study the Noetherian nature of bisingular operators with translation in the space Lp. As corollaries to our general results we obtain criteria for the problem of linear conjugation with translation for analytic functions of two complex variables to be Noetherian.

The theory of one dimensional singular equations with translation has been thoroughly studied. An extensive discussion of the literature on this problem can be found in the survey article [1].

In the present paper we study the Noetherian nature of a bisingular equation with translation $M\phi = f$ in the spaces $L_p(\Gamma_1 \times \Gamma_2)$, where the symbol $M$ designates a bisingular operator with translation (see Sec.3).

The investigation is carried out by means of a modification of a local method. This local method, described in [4], was applied in [2] and [3] to establish conditions for operators of "bilocal type" to be Noetherian. In particular, this method was used in [3] to define and investigate bisingular operators in the spaces $L_p(\Gamma_1 \times \Gamma_2)$. We note that the bisingular operators with translation that we consider are not operators of "bilocal type."

The necessary definition and results on the local method are given in Sec. 1. In Sec. 2 we establish sufficient conditions for a certain class of operators to be partially Noetherian; this class includes the bisingular operators with translation. In Sec. 3 we prove a necessary and sufficient condition for a bisingular operator with translation to be Noetherian. At the same time, as a corollary we obtain criteria for the problem of linear conjugation with translation for an analytic function of two complex variables in the spaces $L_p(\Gamma_1 \times \Gamma_2)$ to be Noetherian.

1. Let $\Gamma_1$ and $\Gamma_2$ be simple closed curves of Lyapunov type, $L_p(\Gamma_i)$ ($i = 1, 2$) the space of complex functions on $\Gamma_i$ with pathpower integrable; $B_p(\Gamma_i)$, $\Lambda_p(\Gamma_i)$, $K_p(\Gamma_i)$ respectively the set of bounded linear operators of local type ([4]), and the completely continuous operators in the space $L_p(\Gamma_i)$.

The symbols

$$B_p(\Gamma_1, \Gamma_2), \Lambda_p^1(\Gamma_1, \Gamma_2), \Lambda_p^2(\Gamma_1, \Gamma_2), \Lambda_p(\Gamma_1, \Gamma_2),$$

$$K_p^1(\Gamma_1, \Gamma_2), K_p^2(\Gamma_1, \Gamma_2)$$

will designate respectively the closure in the norm of the space of bounded linear operators acting in $L_p(\Gamma_1 \times \Gamma_2)$, of the following algebraic tensor products:

$$B_p(\Gamma_1) \otimes B_p(\Gamma_2), \quad B_p(\Gamma_1) \otimes \Lambda_p(\Gamma_2), \quad \Lambda_p(\Gamma_1) \otimes B_p(\Gamma_2), \quad \Lambda_p(\Gamma_1) \otimes \Lambda_p(\Gamma_2), \quad K_p(\Gamma_1) \otimes B_p(\Gamma_2), \quad K_p(\Gamma_1) \otimes K_p(\Gamma_2).$$

The sets $B_p(\Gamma_i, \Gamma_j), \Lambda_p(\Gamma_i, \Gamma_j), \Lambda_p^i(\Gamma_i, \Gamma_j)$ ($i = 1, 2$) are $B$-algebras, $K_p^1(\Gamma_1, \Gamma_2)$ form closed two-sided ideals in the algebras $B_p(\Gamma_1, \Gamma_2)$ and $\Lambda_p^1(\Gamma_1, \Gamma_2)$. The equivalence of two operators $A$ and $B$ with respect to the ideal $K_p^1(\Gamma_1, \Gamma_2)$ will be designated $A \sim B$.

The Noetherian character of the algebras $\Lambda_p(\Gamma_1, \Gamma_2)$ was investigated in [2], and these operators were designated as operators of bilocal type.

**Definition 1.** A bounded operator $A$, acting in the space $L_p(\Gamma_1 \times \Gamma_2)$, is said to be a partially Noetherian operator in the first variable if there exist bounded operators $R_1$ and $R_2$ such that

$$R_1 A - I \in K_p^1(\Gamma_1, \Gamma_2), \quad AR_2 - I \in K_p^1(\Gamma_1, \Gamma_2).$$

The operators $R_1$ and $R_2$ will be called partial regularizers of the operator $A$ in the first variable.

**Definition 2.** Let $t_0 \in \Gamma_1$. The operators $A$ and $B$ from $\Lambda_p^1(\Gamma_1, \Gamma_2)$ are called locally partially equivalent at the point $t_0$ ($A \sim B$), if for any $\varepsilon > 0$ there exists a neighborhood $u$ of the point $t_0$ such that

$$\|(P_u \otimes I)(A - B)\| < \varepsilon,$$

where $P_u$ is the operator representing multiplication by the characteristic function of the neighborhood $u$.

**Definition 3.** A family of operators $\{A_t\}_{t \in \Gamma_1}$ ($A_t \in \Lambda_p^1(\Gamma_1, \Gamma_2)$) is said to be locally continuous if for any $\varepsilon > 0$ and $t_0 \in \Gamma_1$ there exists a neighborhood $u$ of the point $t_0$, such that

$$\|(P_u \otimes I)(A_t - A_{t_0})\| < \varepsilon, \quad \forall t \in u.$$

**PROPOSITION 1.** (The theorem on envelopes, see [4]). Let $\{A_t\}_{t \in \Gamma_1}$ be a locally continuous family of operators from $\Lambda_p^1(\Gamma_1, \Gamma_2)$. Then there exists an operator $A$ from $\Lambda_p^1(\Gamma_1, \Gamma_2)$, unique up to an additive term from the ideal $K_p^1(\Gamma_1, \Gamma_2)$, such that

$$A \sim A_t, \quad \forall t \in \Gamma_1.$$

**Remark.** It is obvious that all of the above definitions and results are valid with respect to the second variable.

2°. Let $\alpha(t)$ be a one-to-one transformation of $\Gamma_1$ onto $\Gamma_1$ satisfying the following conditions:

$\alpha(\alpha(t)) \equiv t$, $\alpha'(t)$ is Holder continuous and $\alpha'(t) \neq 0$ on $\Gamma_1$. We shall say that the function $\alpha(t)$ has type $\langle + \rangle$ ($\langle - \rangle$), if it preserves (changes) the orientation of the contour $\Gamma_1$. The symbol $T$ will designate the translation operator defined as follows: $(T \varphi)(t) = \varphi(\alpha(t))$. Since $\alpha'(t)$ is continuous and $\alpha'(t) \neq 0$, the operator $T$ is continuous in $L_p(\Gamma_1)$ and $T^{-1} = T$.

Let $S$ be the singular operator

$$(S\varphi)(t) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\varphi(t')}{t - t'} dt',$$

then if the operator $T$ has type $\langle + \rangle$ ($\langle - \rangle$), it follows that

$$ST - TS \in K_p(\Gamma_1), \quad (TS + ST) \in K_p(\Gamma_1).$$

In the space of operators $B_p(\Gamma_1, \Gamma_2)$ we consider the set of operators of the form $F + (T \otimes I)G$, where the operators $F$ and $G$ belong to $\Lambda_p^1(\Gamma_1, \Gamma_2)$ and satisfy the relation of local equivalence

$$F \sim_p P_+ \otimes F_1 + P_- \otimes F_1, \quad G \sim_p P_+ \otimes G_1 + P_- \otimes G_1,$$

$$P_\pm = \frac{1}{2}(I \pm S); \quad F_\pm, G_\pm \in B_p(\Gamma_2).$$

**LEMMA 1.** Let $F \in \Lambda_p^1(\Gamma_1, \Gamma_2)$ and satisfy the relation of local equivalence

$$F \sim_p P_+ \otimes F_1 + P_- \otimes F_1.$$

Then the operators $F_1^+ \in F$ and $F_1^- \in F$ are uniquely determined by the operator $F$, and the transformation $t \rightarrow F_1^\pm$ is continuous in the uniform operator topology.

**THEOREM 1.** The operator $F + (T \otimes I)G$, where the operators $F$ and $G$ satisfy the condition (1), while the translation operator $T$ has type $\langle + \rangle$ ($\langle - \rangle$), is partially Noetherian in the first variable if the matrix operators

$$F_1^+, G_1^0, G_1^- F_1^+, F_1^+ G_1^0, G_1^- F_1^+$$

are invertible for all $t \in \Gamma_1$. 

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