ON A CRITERION OF EXACTNESS OF A FINITE FREE COMPLEX

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In this paper we prove the sufficiency of a criterion of exactness of a complex of finitely generated free modules over a commutative ring, which was known earlier for the case of a complex over a Noetherian ring.

Buchsbaum and Eisenbud have obtained in [1] a necessary and sufficient condition for the exactness of a complex of the form

$$0 \to F_k \otimes M \to F_{k-1} \otimes M \to \ldots \to F_1 \otimes M \to F_0 \otimes M.$$  (1)

Here $F_i$ and $M$ are finitely generated modules over a commutative Noetherian ring and, moreover, all $F_i$ are free. Eagon and Northcott have suggested in [2] a generalization of the Buchsbaum–Eisenbud condition and have shown that the generalized condition is necessary for the exactness of complex (1) over an arbitrary commutative ring.

In the present paper it is proved that the Eagon–Northcott condition is also sufficient for the exactness of complex (1). By the same token we obtain criteria for the exactness of a finite free complex over an arbitrary commutative ring.

Let $R$ be a commutative ring, $E$ be an $R$-module, and $I \subseteq R$ be an ideal. A sequence of elements $x_0 = 0, x_1, \ldots, x_k \in R$ is called regular on $E$ if $x_i$ is not a zero-divisor on $E/(x_0, \ldots, x_{i-1})E$ and $E/(x_0, \ldots, x_{i-1})E \cong E/(x_0, \ldots, x_i)E$ for $1 \leq i \leq k$. Here $(x_0, \ldots, x_i)$ denote the ideal generated in $R$ by the elements $x_0, \ldots, x_i$.

Definition 1. $c_{gr}(I, E)$ (the degree of $I$ over $E$) is the supremum of the lengths of regular sequences on $E$ which are contained in $I$.

Let $c_h = c_{gr}(I, E) (I[X_1, \ldots, X_k], E[X_1, \ldots, X_k])$, where $X_1, \ldots, X_k$ are independent variables. It is clear that $c_1 \leq c_2 \leq \ldots$.

Definition 2. $k_{gr}(I, E)$ (the polynomial degree of $I$ over $E$) = $\lim_{k \to \infty} c_h$ is a natural number or $\infty$.

Let $\phi : F \to G$ be a homomorphism of finitely generated free $R$-modules and $M$ be an $R$-module. Let us put $\text{rank } (\phi, M) = \sup \{n: \text{the ideal generated by the $n$-th order minors of the matrix of } \phi \text{ is not contained in } \text{Ann } M\}$. Let $I(\phi, M)$ be the ideal generated by the $n$-th order minors of the matrix of $\phi$, where $n = \text{rank } (\phi, M)$.

THEOREM. Let $F : 0 \to F_n \to F_{n-1} \to \ldots \to F_0$ be a complex of finitely generated free $R$-modules and $M$ be a finitely generated $R$-module. Let us assume that $\phi_k \otimes 1_M \neq 0$ for every $k \geq 1$ and

(1) $\text{rank } (\phi_k, M) + \text{rank } (\phi_{k+1}, M) = \text{rank } F_k$.

(2) $k_{gr}(I(\phi_k, M), M) \geq k$ or $I(\phi_k, M) = R$.

Then $F \otimes M$ is exact.

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Remark. By virtue of [2], the condition formulated above is also necessary for the exactness of the complex \( F \otimes M \).

Let us now state, for ease of reference, some results which we shall need.

**Lemma A** ([1], Lemma 1). Let \( \varphi : F \to G \) be a mapping of finitely generated free \( R \)-modules. Then \( \text{Coker} \varphi \) is projective and has a definite rank if and only if \( I(\varphi) = R \).

**Lemma B** ([3], Propositions 3 and 4). Let \( I \) be a finitely generated ideal in \( R \) and \( M \) be an \( R \)-module such that \( IM \neq M \). Then k. gr. \( (I, M) < \infty \) and k. gr. \( (I, M) = 1 \) is equal to the smallest integer \( n \) such that \( \text{Ext}^n(R/I, M_M) \neq 0 \), where \( M_M \) is the sum of a countable number of copies of \( E \).

This result due to Barger shows the notion of polynomial degree as near to the analogous notion of degree for Noetherian rings.

Since \( F \otimes_R M \) is exact if and only if \( F [X_1, \ldots, X_p] \otimes_R R[X_1, \ldots, X_p] \otimes M \) is exact for some \( p > 0 \), we may assume that c. gr. \( (I(\varphi_k), M) \geq k \geq 1 \). Hence, \( \text{Ann} (I(\varphi_k), M) \subset \text{Ann} M \). Consequently, the first condition of the theorem is preserved under localization. Since it follows from the definitions that the second condition is also preserved, we may assume \( R \) to be local. But in this case, since \( M \) is finitely generated, we have \( I(\varphi_k, M) \cdot M = M \) if \( I(\varphi_k, M) = R \), i.e., we can apply Lemma B. Further, the complex \( F \otimes M \) can be considered over the ring \( R/\text{Ann} M \), i.e., we may assume that \( \text{Ann} M = 0 \). In this case, rank \( I(\varphi_k, M) = \text{rank} \varphi_k \) for all \( k \).

We shall now prove the theorem by induction over the length of \( F \). Case \( n = 1 \). We have: \( 0 \to F_1 \to F_0 \), and c. gr. \( (I(\varphi_1), M) \geq 1 \). Let \( r \in I(\varphi_1) \) not be a zero-divisor on \( M \). Since \( \text{Ann} M = 0 \), \( r \) is not a zero-divisor on \( R \). Let us pass from \( R \) to \( R_r \). According to Lemma A, \( \text{Coker} (\varphi_1)_R \) is projective. Let us show that \( (\varphi_1)_R \) is an embedding onto a direct summand. If \( (\varphi_1)_R \) is not an embedding, then there exists a prime ideal \( p \subset \mathbb{R} \) such that \( (\varphi_1)_p \) is not an embedding. But \( \text{Coker} (\varphi_1)_R \) is free. Consequently, \( \text{Im} (\varphi_1)_p \) and ker \( (\varphi_1)_p \) are free, and since \( \text{rank} (\varphi_1)_p = r \) = \( \text{rank} F_1 = \text{rank} (\varphi_1)_p \), we have \( \text{ker} (\varphi_1)_p = 0 \). (The first two equalities follow from the fact that \( \text{Coker} (\varphi_1)_R \) has a fixed rank according to Lemma A.) Hence, \( (\varphi_1)_R \) is an embedding. Since \( \text{Coker} (\varphi_1)_R \) is projective, \( (\varphi_1)_R \) is an embedding onto a direct summand and \( (\varphi_1)_R \otimes_{R_{(p)}} 1 \) is also an embedding. In the diagram

\[
\begin{array}{ccc}
(F_1 \otimes M)_{(p)} & \overset{(\varphi_1 \otimes 1_M)_{(p)}}{\longrightarrow} & (F_0 \otimes M)_{(p)} \\
\downarrow & & \downarrow \\
F_1 \otimes M & \overset{(\varphi_1 \otimes 1_M)}{\longrightarrow} & F_0 \otimes M
\end{array}
\]

the upper arrow is an embedding, as proved, and the vertical arrows are embeddings since \( r \) is not a zero-divisor on \( M \). This means that the lower horizontal arrow is also an embedding. The theorem is proved for the case \( n = 1 \).

Let the theorem be valid for all complexes of lengths \( n < k \) and let us prove it for complexes of length \( k \). We have

\[
F: 0 \to F_k \to F_{k-1} \to \ldots \to F_2 \to F_1 \to F_0,
\]

\[
F': 0 \to F_k \to F_{k-1} \to \ldots \to F_2 \to F_1 \to K \to 0,
\]

where \( K = \text{Coker} \varphi_{k-1} \) and \( 0 \to K \to F_k \), \( \varphi_{k-1} \) being the composition \( K = F_k / \text{Im} \varphi_{k-1} \). According to the induction hypothesis, \( F' \otimes M \) is exact. It is necessary for us to prove that \( \varphi_k \otimes 1_M \) is an embedding.

The remaining part of this paper is devoted to this proof.

If \( I(\varphi_k) = R \), then, according to Lemma A, \( \text{Coker} \varphi_k \) is projective and by localization the whole thing is reduced to a complex of lesser length; this means we can assume that \( I(\varphi_k) \neq R \).

**Lemma 1.** c. gr. \( (I(\varphi_k), K \otimes M) \geq 1 \).

**Proof.** Let \( K_i = \text{Im} (\varphi_i \otimes M) = \ker (\varphi_{i-1} \otimes M) \) \((i \geq 2)\), \( K_1 = K \otimes M \). Let us prove by induction over \( i \) (in the reverse direction) that c. gr. \( (I(\varphi_i), K_i) \geq 1 \). For \( i = k \) we have \( 0 \to F_k \otimes M \to K_k \to 0 \) and k. gr. \( (I(\varphi_k), K_k) = \text{c. gr.} (I(\varphi_k), F_k \otimes M) \geq k \). For \( i = k-1 \) we have \( 0 \to K_{i-1} \otimes M \to K_i \to 0 \) is exact. Hence, the sequence \( 0 \to K_{i-1} \otimes M \to K_i \to 0 \) is also exact. Now a standard