DECOMPOSABILITY OF MODULES INTO A DIRECT SUM OF IDEALS

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It is shown that each left R module is isomorphic to a direct sum of left ideals of the ring R if and only if R is quasi-Frobenius and generalized uniserial.

The definition of a quasi-Frobenius ring and various characterizations of this class are well known (see, e.g., [1, p. 395, (58.5)]). A left (right) Artinian ring is called left (right) generalized uniserial if the set of left (right) ideals of this ring contained in a left (right) ideal generated by a primitive idempotent is linearly ordered relative to inclusion. A ring is called generalized uniserial if it is both left and right generalized uniserial.

Our main result:

THEOREM 1. Each left R module is isomorphic to a direct sum of left ideals of the ring R if and only if R is quasi-Frobenius and generalized uniserial.

Nonzero elements α₁, ..., αₙ of a ring R are called absolutely independent if for each i there exists an element uᵢ ∈ R such that

\[ uᵢαⱼ = \begin{cases} αᵢ, & \text{if } i = j, \\ 0, & \text{if } i ≠ j. \end{cases} \]

If \( X ⊆ R \), we put

\[ t(X) = \{ y; y ∈ R, yx = 0 \text{ for all } x ∈ X \} \]

and

\[ r(X) = \{ y; y ∈ R, xy = 0 \text{ for all } x ∈ X \}. \]

For the proof of our main result we will need

THEOREM 2. Each cyclic left R module is isomorphic to a direct sum of left ideals of the ring R if and only if each left ideal of R is the left annihilator of some finite set of absolutely independent elements of R.

Theorem 2 can be viewed as a description of the class of rings over which each cyclic left R module is isomorphic to a direct sum of left ideals. T. S. Tol'skaya [2, p. 190] gave an example of a ring over which each cyclic left module is isomorphic to a large left
ideal. This ring is left but not right Artinian. Therefore, it follows from the results of Rutter [3, p. 535, Theorem 5] that not all finitely generated left modules over this ring are isomorphic to a direct sum of left ideals. On the other hand, the author has no examples showing that the requirement of "decomposability into a direct sum of ideals" of any left module is stronger than the same requirement for finitely generated left modules. In this connection, it is interesting to note that the latter requirement implies the flatness of any injective left module [3, p. 534, Lemma 2]. We also mention that, in view of the results of Faith and Walker [4, p. 217, Corollary 5.10], a ring over which both left and right cyclic modules are isomorphic to a direct sum of ideals must be quasi-Frobenius, and the fact that it is uniserial follows from Lemma 5 proved below.

As usual, we assume that a ring has a unity and that the modules considered are unital.

Proof of Theorem 2. Assume that for each left ideal $L$ of the ring $R$ there exists an isomorphism

$$
\chi: R/L \rightarrow I_1 \oplus \cdots \oplus I_n
$$

where $I_1, \ldots, I_n$ are left ideals of $R$. The number of summands is finite, since $R/L$ is cyclic. Let

$$
\chi (1 + L) = (a_1, \ldots, a_n)
$$

and

$$
u_i + L = \chi^{-1} (0, \ldots, a_i, \ldots, 0),
$$

where $a_i \in I_i$. Then

$$(0, \ldots, a_i, \ldots, 0) = \chi (u_i + L) = \chi (u_i (1 + L)) = (u_i a_1, \ldots, u_i a_n).$$

Thus,

$$u_i a_j = \begin{cases} a_i , & \text{if } i = j, \\ 0 , & \text{if } i \neq j, \end{cases}
$$

i.e., the elements $a_1, \ldots, a_n$ are absolutely independent. On the other hand, if $x \in L$, then $\chi (x (1 + L)) = 0$, hence, $x a_i = 0$ for all $i$, i.e., $L \subseteq \bigcap \{a_i\}$. Conversely, if $x \in \bigcap \{a_i\}$, then

$$x (1 + L) = \chi^{-1} (x (a_1, \ldots, a_n)) = 0 + L,$$

i.e., $x \in L$. Thus, $L = \bigcap \{a_i\}$, as required. Now assume that the stated property of left ideals holds. Suppose $L = \bigcap \{a_i\}$, where $a_1, \ldots, a_n$ are absolutely independent, and consider the homomorphism

$$
\varphi: R \rightarrow Ra_1 \oplus \cdots \oplus Ra_n,
$$

where

$$\varphi (1) = (a_1, \ldots, a_n).$$

Since

$$\varphi (u_i) = (u_i a_1, \ldots, u_i a_n) = (0, \ldots, 0, a_i, 0, \ldots, 0),$$

where the $u_i$ are the elements mentioned in the definition of absolute independence, $\varphi$ is surjective. Consequently,

$$Ra_1 \oplus \cdots \oplus Ra_n \cong R/\ker \varphi.$$

If $x \in L$, then

$$\varphi (x) = (xa_1, \ldots, xa_n) = (0, \ldots, 0),$$

i.e., $x \in \ker \varphi$. If $x \in \ker \varphi$, then

$$(0, \ldots, 0) = \varphi (x) = (xa_1, \ldots, xa_n),$$

i.e., $x \in \bigcap \{a_i\} = L$. Thus, $L = \ker \varphi$.

We will now prove several lemmas, using, for the sake of brevity, the following notation: