LAGRANGE INTERPOLATION POLYNOMIALS AND ORTHOGONAL FOURIER–JACOBI SERIES

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UDC 517

Let \( \alpha > -1 \) and \( \beta > -1 \). Then a function \( f(x) \), continuous on the segment \([-1; 1]\), exists such that the sequence of Lagrange interpolation polynomials constructed from the roots of Jacobi polynomials diverges almost everywhere on \([-1; 1]\), and, at the same time, the Fourier–Jacobi series of function \( f(x) \) converges uniformly to \( f(x) \) on any segment \([a; b] \subset (1; 1)\).

Let \( \alpha > -1, \beta > -1 \) and \( \{P_n^{(\alpha, \beta)}(x)\} \) be a sequence of Jacobi polynomials forming on the segment \([-1; 1]\) an orthonormal system of weight \( w(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \). We call the matrix \( \mathbf{M}^{(\alpha, \beta)} = \{x_k, n\}, 1 \leq k \leq n, \ n = 1, 2, 3, \ldots, \) of the interpolation nodes a Jacobi matrix if its \( n \)-th row consists of the roots of the polynomial \( P_n^{(\alpha, \beta)}(z) \).

For any real function \( f(x) \), continuous on \([-1; 1]\), \( f(x) \in C([-1; 1]) \), we set

\[
L_n(\mathbf{M}^{(\alpha, \beta)}, f, x) = \sum_{k=1}^{n} f(x_k, n) I_{k, n}(\mathbf{M}^{(\alpha, \beta)}, x), \ n = 1, 2, 3, \ldots,
\]

and

\[
S^{(\alpha, \beta)}(f, x) = \sum_{k=0}^{\infty} c_k P_k^{(\alpha, \beta)}(x), \quad (1)
\]

where

\[
I_{k, n}(\mathbf{M}^{(\alpha, \beta)}, x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(x_k, n)}(x - x_k, n),
\]

and

\[
c_k = \int_{-1}^{1} (1 - t)^{\alpha}(1 + t)^{\beta} f(t) P_k^{(\alpha, \beta)}(t) \, dt.
\]

One of the questions in function approximation theory is the determination of the equiconvergence of the two infinite processes of Lagrange and of Fourier–Jacobi.

For \( \alpha = \beta = -\frac{1}{2} \), Erdős and Grünwald [1] proved the existence of a continuous function \( f(x) = f(\cos \theta) \) whose Fourier series converges uniformly but, at the same time, the corresponding sequence of Lagrange polynomials diverges everywhere.

The aim of this note is to prove the validity of the following statement.

**THEOREM 1.** Let \( \alpha > -1, \beta > -1 \), and \( \mathbf{M}^{(\alpha, \beta)} \) be a Jacobi matrix. Then a function \( f(x) \in C([-1; 1]) \) exists such that the sequence \( \{L_n(\mathbf{M}^{(\alpha, \beta)}, f, x)\} \) of Lagrange interpolation polynomials diverges almost everywhere on \([-1; 1]\).
almost everywhere on \([-1; 1]\) and, at the same time, the Fourier–Jacobi series \(S(\alpha, \beta)(f, x)\) converges uniformly on the segment \([-1 + h; 1 - h]\), for any number \(h, 0 < h < 1\).

All the arguments which follow are carried out for arbitrary fixed numbers \(\alpha\) and \(\beta\), \(\min \{\alpha, \beta\} > -1\).

**Lemma 1.** The estimate

\[
P_n^{(\alpha, \beta)}(x) = O_x(1),
\]

is valid at each point \(x \in (-1; 1)\) for the orthogonal system \({P_n^{(\alpha, \beta)}(x)}\) of Jacobi polynomials; moreover, this estimate is uniform on each segment \([a, b] \subset (-1; 1)\). If, however, the function \(f(x) \in C((-1; 1))\) satisfies a Dini–Lipschitz condition of order \(\delta, \delta > 1\), on the segment \([a, b] \subset (-1; 1)\), then expansion (1) of function \(f(x)\) in the Jacobi polynomials converges uniformly on \([a; b]\).

(See [2, pp. 35, 52, 54, and Theorems 1.5.4 and 1.5.6] for the definition of the Dini–Lipschitz condition and for the proof of Lemma 1.)

By \(S_m^{(\alpha, \beta)}(f, x)\) we denote a partial sum of series (1) and we consider the "roof-function"

\[
\lambda_\delta(x, x_0) = \begin{cases} 
1, & \text{if } x = x_0, \\
0, & \text{if } x \in (x_0 - \delta; x_0 + \delta), \\
\text{is linear on } [x_0 - \delta; x_0] \text{ and } [x_0; x_0 + \delta].
\end{cases}
\]

(2)

where the point \(x_0 \in (-1; 1)\) and \(0 < \delta < 1 - |x_0|\).

**Lemma 2.** Let a positive number \(h, 0 < h < 1\), be given. Then:

a) A constant \(c_1\) exists, depending only on \(h\), such that the inequalities

\[
|S_m^{(\alpha, \beta)}(\lambda_\delta, x)| < c_1,
\]

are valid for all \(m, \delta \in (0; h/2], x_0 \in [-1 + h; 1 - h]\) and \(x \in [-1 + h/2; 1 - h/2]\);

b) for any preassigned positive numbers \(\epsilon\) and \(\eta\) there exists a positive number \(\delta_0\) such that

\[
|S_m^{(\alpha, \beta)}(\lambda_\delta, x)| < \epsilon, \quad \forall m,
\]

if only \(x_0 \in [-1 + h; 1 - h], x \in [-1 + h/2; 1 - h/2], 0 < \eta < |x - x_0| < 2 - 3/2h\), and \(0 < \delta < \delta_0\).

**Proof.** If \(1 + h/2 \leq x, t \leq 1 - h/2\), then by virtue of Lemma 1 a constant \(c_2\) exists such that

\[
|P_n^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(t)| \leq c_2 |x - t|,
\]

In addition, denoting the leading coefficient of the polynomial \(P_n^{(\alpha, \beta)}(x)\) by \(a_n\), we have \(0 < a_n/a_{n+1} \leq 1\) [2, p. 36]. Hence from Eq. (2), having applied the Christoffel–Darboux formula [2, p. 34], for all \(m, \delta \in (0; h/2], x_0 \in [-1 + h; 1 - h]\) and \(x \in [-1 + h/2; 1 - h/2]\) we obtain

\[
|S_m^{(\alpha, \beta)}(\lambda_\delta, x) - \lambda_\delta(x, x_0)| \leq \sum_{k=0}^{m} \frac{1}{|x - x_0|^k} \left(1 - t\right)^{\alpha} (1 + t)^{\beta} |\lambda_\delta(t, x_0) - \lambda_\delta(x, x_0)|.
\]

Next, denote the leading coefficient of the polynomial \(P_n^{(\alpha, \beta)}(x)\) by \(a_n\), then for all \(m, \delta \in (0; h/2], x_0 \in [-1 + h; 1 - h]\) and \(x \in [-1 + h/2; 1 - h/2]\) we obtain

\[
|S_m^{(\alpha, \beta)}(\lambda_\delta, x)| \leq \sum_{k=0}^{m} \frac{|x - x_0|^k}{|x - x_0|^k} \left(1 - t\right)^{\alpha} (1 + t)^{\beta} |\lambda_\delta(t, x_0) - \lambda_\delta(x, x_0)|.
\]

Hence follows the first assertion of the lemma.

Further,

\[
|x - x_0| > \eta \quad \text{and} \quad \sum_{k=0}^{m} \frac{P_n^{(\alpha, \beta)}(t) P_n^{(\alpha, \beta)}(x)}{t - x} \leq c_2/|t - x|,
\]

therefore, by virtue of Eq. (2), we have

\[
|S_m^{(\alpha, \beta)}(\lambda_\delta, x)| \leq \sum_{k=0}^{m} \frac{|x - x_0|^k}{|x - x_0|^k} \left(1 - t\right)^{\alpha} (1 + t)^{\beta} \frac{1}{|t - x|} dt \leq c_2 \sum_{k=0}^{m} \frac{|x - x_0|^k}{|x - x_0|^k} \left(1 - t\right)^{\alpha} (1 + t)^{\beta} dt \leq c_2.
\]