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LITERATURE CITED


NORMED SPACES WHICH SATISFY APOLLONIUS' THEOREM

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It is proved that a normed space is a Hilbert space if it possesses the property: The geometric locus of the points, for which the ratio of the distances to two given points is constant, is a sphere.

In May 1967, at the Second All-Union Symposium on Integral Geometry, held in Petrozavodsk, S. B. Stechkin posed the following problem: Is a Banach space a Hilbert space if it possesses the property: a) The geometric locus of the points, for which the ratio of the distances to two given points is constant, is a sphere [1]?

To this question we give an affirmative answer.

THEOREM. A normed linear space \( B \) which possesses property a) is a Hilbert space.

For the proof of the theorem it is sufficient to show that in \( B \) one can introduce an inner product which induces the given norm. First we prove this for a real \( B \). To this end, by virtue of Ficken's theorem (see [2, p. 193]), it is sufficient to prove that from property a) the following lemma is obtained.

LEMMA. Let \( a, b \in B, \|a\| = \|b\| = 1 \). Then for any real number we have the equality

\[ \|b + xa\| = \|ab + a\|. \]

Proof. Let \( 0 < a < 1, \|a\| = 1 \). Then

\[ \|a - \frac{a}{a}a\| = \|a - \frac{a}{a}a\| = a. \] (1)

By virtue of property a)
where $S(c, R)$ is the sphere in $B$ with center at $c$ and radius $R > 0$.

Let us prove that $c = 0$. We assume the opposite, i.e., that $c \neq 0$. By virtue of (1), we have

$$\pm a \in S(c, R) \iff \|c \pm a\| = R. \quad (2)$$

Let $x \in S(0, R)$. Then $x + c \in S(c, R)$, and consequently,

$$\|x + c - \alpha a\| = \alpha \|x + c - \frac{1}{\alpha} a\| = \|x + \alpha c - a\|.\quad (3)$$

But

$$\bigcup_{x \in S(0, R)} (x + c - \alpha a) = S(c - \alpha a, R),$$

$$\bigcup_{x \in S(0, R)} (\alpha x + \alpha c - x) = S(\alpha c - a, \alpha R).$$

The spheres $S(c - \alpha a, R)$ and $S(\alpha c - a, \alpha R)$ are homothetic (the homothety center is $p = -(1 + \alpha) a$, the coefficient is $k = \alpha$): $\|p - c \div \alpha a\| = \|c - a\| = R$ (by virtue of (2)). Thus, $p \in S(c - \alpha a, R)$. In addition,

$$S(\alpha c - a, \alpha R) \subset \{x \|x - c + \alpha a\| \leq R\}.$$  

Clearly, $c$ and $\alpha$ are linearly independent. Indeed, let us assume the opposite, i.e., that $c = \lambda \alpha$. Then

$$\|\lambda a - a\| = \|\lambda a + a\| = R.$$  

From here $|\lambda - 1| = |\lambda + 1| \iff \lambda = 0$, which contradicts the assumption $c \neq 0$.

Let $B_2$ be the two-dimensional subspace spanned by $\alpha$ and $c$. Then

$$S(0, R) \cap B_2 = S_2(0, R), \quad S(c - \alpha a, R) \cap B_2 = S_2(c - \alpha a, R),$$

$$S(\alpha c - a, \alpha R) \cap B_2 = S_2(\alpha c - a, \alpha R),$$

where $S_2(\ast)$ are the appropriate spheres in $B_2$.

In the sequel we shall denote the line

$$\{x \mid x = a + (b - a) t, \ -\infty \leq t \leq \infty\} = (a, b),$$

the segment

$$\{x \mid x = tb + (1 - t)a, \ 0 \leq t \leq 1\} = [a, b].$$

Equation (3) has the following geometric meaning: Let $H_{p, a}$ be a homothety in $B_2$ with center at $p$ and coefficient $\alpha, m \in S_2(c - \alpha a, R) \{0 \cup p\}, A(m) = Rm/\|m\|$. Then the lines $(m, A(m), A(H_{p, a}(m)))$ are parallel and

$$A(m), A(H_{p, a}(m)) \in S_2(0, R), H_{p, a}(m) \in S_2(\alpha c - a, \alpha R).$$

Let us prove that the line $(-a, a)$ is not supporting for $S_2(c - \alpha a, R)$. We assume the opposite. Then $S_2(c - \alpha a, R) \cap (-a, a) = [p_1, p_2]$ is a segment with the endpoints $p_1$ and $p_2$ (possibly, $p_1 = p_2 = p$). If $p_1 \neq p_2$, then we take $m \in S_2(c - \alpha a, R) \{p_1, p_2\}$. The points $m \mapsto p_1, m$ are situated on the given side of $p_1$. Then the line

$$(p, m) \rightarrow (-a, a), \quad A(m) \rightarrow A(p),$$

$$A(H_{p, a}(m)) \rightarrow A(p), \quad A(m) \neq A(H_{p, a}(m)).$$

Therefore the line $(A(m), A(H_{p, a}(m)))$ converges to the supporting line to $S_2(0, R)$ at the point $A(p)$. But

$$(A(m), A(H_{p, a}(m))) \parallel (p, m).$$

Thus, $(-a, a)$ is a supporting line to $S_2(0, R)$, which is not possible since this line passes through the center $0$. If $p_1 = p_2 = p$, then the argument is similar. In this case, when $m \rightarrow p$, then on one hand the line $(p, m) \rightarrow Z_1(p)$, on the other hand $(p, m) \rightarrow Z_2(p)$,