GENERALIZED RESOLVENTS OF CLOSED DISSIPATIVE OPERATORS
IN A HILBERT SPACE WITH A \( \psi \) METRIC

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UDC 513.8

Generalized resolvents of a closed \( \psi \)-dissipative operator in a Hilbert space with a \( \psi \) metric are defined and studied; in particular, a formula of generalized resolvents of such an operator is obtained.

1. With regard to its content, this note is a direct continuation of the author's papers [1] and [2], and of Nikonov's and the author's paper [3]; it is devoted to a description of a set of generalized resolvents of a closed \( \psi \)-dissipative operator acting in a Hilbert space with a \( \psi \) metric in terms of this operator and of the space in which it acts. The obtained formula of generalized resolvents extends (and its proof strongly relies on) the formula of Shtraus [4] for generalized resolvents of a closed symmetric operator in ordinary Hilbert space.

2. In this paper we shall use the following abbreviations: c. \( \psi \)-d. (closed \( \psi \)-dissipative), c. \( \psi \)-s. (closed \( \psi \)-symmetrical), c. m. \( \psi \)-d. (closed maximal \( \psi \)-dissipative), \( \psi \)-s.-a. (\( \psi \)-self-adjoint), g. r. (generalized resolvent), c. n. (closed nonstretching), c. i. (closed isometric), and for \( \psi = I \) (i.e., in the case of an ordinary Hilbert space) we shall not write the letter I in any of these abbreviations. The definitions of the concepts used in this note can be found in [1-3].

We shall use the following notations: \( C_+ \) is an upper or lower open half-plane; \( C_+^{(1)} = \{ z \in C_+: \varepsilon < |\arg z| < \pi - \varepsilon, \quad 0 < \varepsilon < \pi /2 \} \); \( B \) is a c. \( \psi \)-d. operator in a Hilbert space \( \mathcal{H} \) with a \( \psi \) metric (or, what amounts to the same, in a \( \psi \) space \( \mathcal{H} \)); \( A = B \psi \) is a c. d. operator with \( \mathcal{D}(A) = \mathcal{D}(B) \), \( z \in C_+ \), \( \mathcal{M}_z = \mathcal{R}(B - z\psi) = \mathcal{R}(A - zI) \), \( \mathcal{M}_z^* = \mathcal{R}(B - z\psi) = \mathcal{R}(A - zI) \), \( \mathcal{M}_z = \mathcal{H} \cap \mathcal{M}_z \), \( W_z = (B - z\psi) (B - z\psi)^* = (A - zI) (A - zI)^{-1} \) a c. n. operator without nonzero fixed elements; \( \hat{W}_z = W_z P_z \), where \( P_z \) is an orthogonal projection operator from \( \mathcal{H} \) to \( \mathcal{M}_z \); \( T_z = (I - \hat{W}_z \hat{W}_z^* \mathcal{H}) \) is a nonnegative c. n. operator with \( \mathcal{D}(T_z) = \mathcal{H} \). Let us note that \( W_z \) is a c. i. operator, and then \( T_z \) will be an orthogonal projection operator from \( \mathcal{H} \) to \( \mathcal{M}_z \) if \( B \) is a c. \( \psi \)-s. operator (or, what amounts to the
same, A is a c. s. operator). Next we need the concepts (see [2] and [3]) of an operator $X_\mathcal{F}$ from $\mathcal{F}(T)$ to $\mathcal{F}(T)$, or of an operator $V$ from $\mathcal{F}(T)$ to $\mathcal{F}(T)$ said to be forbidden or allowed (respectively) with respect to the operator $B$ (or, what amounts to the same, the operator $A$). We shall also use (see [3, Theorem 4]) the description of all c. $\mathcal{F}$-d. (and, in particular, c. m. $\mathcal{F}$-d.) extensions $B = B_\mathcal{F}$ of the operator $B$ in terms of the original operators $\mathcal{F}$ and $B$ and of operators $\overline{V}$ that are allowed with respect to $B$. In this connection let us note the obvious relation $B_\mathcal{F} = (A, \mathcal{F})$. Let $F$ be a closed operator in the $\mathcal{F}$ space $\mathcal{F}$. Let us write $G = F/\{g \in \mathcal{D}(F) : [Fg, g] = [f, Fg] \text{ for any } f \in \mathcal{D}(F)\}$; it is evident that $G$ is a c. $\mathcal{F}$-s operator that will be henceforth called a c. $\mathcal{F}$-s. kernel of the operator $F$ and denoted by $G_\mathcal{F}$=sk($F$). For $\mathcal{F} = I$, we arrive (see [1]) at the concept of c. s. kernel $G = \text{sk}(F)$ of the operator $F$. It is evident that $G = \mathcal{F} - \text{sk}(F) = F$ only if $F$ is a c. $\mathcal{F}$-s. operator.

Now let: $G = \mathcal{F} - \text{sk}(F), S = \text{sk}(A)$; it is evident that $S = G\mathcal{F}$. From Theorem 4 of [3] it follows, firstly, that $B = A\mathcal{F} = (S, V_A)\mathcal{F} = G_\mathcal{F} V_A$, where $V_B = V_A = W_z \mathcal{F}' \cap \mathcal{F}(B - z\mathcal{F})$ is a c. n. operator from $\mathcal{F}' \cap \mathcal{F}(B - z\mathcal{F}) = \mathcal{F} \cap \mathcal{F}(S - zI)$ to $\mathcal{F}' \cap \mathcal{F}(G - z\mathcal{F}) = \mathcal{F} \cap \mathcal{F}(S - zI))$ that is allowed with respect to $G$ (or, what amounts to the same, with respect to $S$); secondly, we obtain the following theorem.

**THEOREM 1.** If $B$ is a c. $\mathcal{F}$-d. operator in a $\mathcal{F}$ space $\mathcal{H}$ that has any domain of definition $\mathcal{D}(B)$, then for any $z \in C_-$ there exists a one-to-one correspondence between the totality of allowed, with respect to $G = \mathcal{F} - \text{sk}(F)$, c. n. operators $U$ from $\mathcal{F} \cap \mathcal{F}(G - z\mathcal{F})$ to $\mathcal{F} \cap \mathcal{F}(G - z\mathcal{F})$ that are extensions of the operator $V_B$, and the totality of c. $\mathcal{F}$-d. extensions $\mathcal{F}$ of the operator $B$, with $D (B) = D (G) = \mathcal{F} (U - I) \mathcal{D}(U)$, $B (f + \mathcal{F} \mathcal{D}(U) - \mathcal{F}(U)) = G f + (a U - z I) v$, $(U \in D (G), v \in \mathcal{D}(U))$, $U = (B - z\mathcal{F})^{-1}$. The operator $\mathcal{F}$ is c. m. $\mathcal{F}$-d. if and only if $D (U) = \mathcal{F}'$.

Let us specify the connection between Theorem 4 of [3] and Theorem 1. Let us denote by $L_\mathcal{F}(B)$ (or $L_\mathcal{F}(G)$) the totality of c. n. operators from $\mathcal{F}$ to $\mathcal{F}'$ that are extensions of the operator $V_B$, and the totality of c. $\mathcal{F}$-d. extensions $\mathcal{F}$ of the operator $B$, with $D (B) = D (G) = \mathcal{F} (U - I) \mathcal{D}(U)$. Let us denote by $\mathcal{F}$ $\mathcal{F}$ (or $\mathcal{F}$ $\mathcal{F}$) the totality of operators in $\mathcal{F}$ $\mathcal{F}$ defined on the entire $\mathcal{F}$ $\mathcal{F}$ (or $\mathcal{F}$ $\mathcal{F}$). From [1] we obtain the following

**LEMMA 1.** Between the set of all operators $\mathcal{F}$ in $\mathcal{F}$ (or $\mathcal{F}$ $\mathcal{F}$) for which $V_B \subseteq \mathcal{F}$ and the set of all operators $\mathcal{F}$ in $\mathcal{F}$ (or $\mathcal{F}$ $\mathcal{F}$) there exists a one-to-one correspondence, with $D (U) = D (V_B) \cap \mathcal{D}(U)$, $U (\mathcal{F} = \mathcal{F} \cap \mathcal{F}(G - z\mathcal{F})$, and we have the relation $G_\mathcal{F} U = B_\mathcal{F}$.

Remark 1. From Crandall's paper [5] it also follows that the definition of the operator classes occurring in the formulation of Lemma 1 remains valid (apart from the last equation) also without requiring that the corresponding operator be allowed.

3. Now let us consider the g. r. of the original c. $\mathcal{F}$-d. operator $B$ in the $\mathcal{F}$-space $\mathcal{H}$. Let $\mathcal{H}$ be a Hilbert space. Let us construct a new $\mathcal{F}$ space $\mathcal{H} = \mathcal{H} \cap \mathcal{F}$ that contains the original $\mathcal{F}$ space $\mathcal{H}$ in the above sense. It is evident that in $\mathcal{H}$ the operator $B$ is a c. $\mathcal{F}$-d. operator, and hence, it always has c. m. $\mathcal{F}$-d. extensions $\mathcal{F}$. Following Krein and Langer [6], we shall say that these extensions are regular.

**Definition 1.** The g. r. of a c. $\mathcal{F}$-d. operator $B$ in a $\mathcal{F}$ space $\mathcal{H}$ corresponding to a regular c. m. $\mathcal{F}$-d. extension $\mathcal{F}$ of the operator $B$ that goes to a $\mathcal{F}$ space $\mathcal{H}$ which has a nonempty set $\rho (B) \cap C_-$ is specified by an operator-valued function of $z \in \rho (B) \cap C_-$ defined by the formula $R (z) = P R (z) \mathcal{H}$, where $P$ is an orthogonal projection operator (or, what amounts to the same, a $\mathcal{F}$ orthogonal projection operator) from $\mathcal{H}$ to $\mathcal{H}$, and $R (z) = (B - z I)^{-1}$ is the ordinary resolvent of the operator $B$ in $\mathcal{H}$.

Remark 2. Let us note that we are considering only regular $\mathcal{F}$-s. a. extensions $\mathcal{F}$ which exist only if $\dim (\mathcal{F} \cap \mathcal{H}) = \dim (\mathcal{F} \cap \mathcal{H})$ for any $z \in C_ \cup C_+$ with a nonempty set $\rho (B)$ if $B$ is a c. $\mathcal{F}$-c. operator, and $z, \bar{z} \in \rho (B) \cap (C_ \cup C_+)$; in this case it is easy to prove...