THE COMPLEXITY OF THE REALIZATION OF ALMOST ALL FUNCTIONS OF THE ALGEBRA OF LOGIC BY THE METHOD OF CASCADES

I. D. Denev

We give an asymptotic bound for the complexity of almost all functions for the method of cascades.

The method of cascades is a refinement of Shannon's method for the synthesis of systems. The method of cascades is described in [1, 2]. The upper bound for Shannon's function $L(n)$ for the method of cascades coincides with the upper bound for Shannon's function for Shannon's method. It follows from [2] that

$$L(n) \leq C_n \frac{2^n}{n} (1 + o(1)),$$

where the approximate form of the graph of the coefficient $C_n$ is shown in Fig. 1.

A good lower bound for $L(n)$ was obtained recently by Kuznetsov [3]. For all values of $n - \log n$ sufficiently distant from an arbitrary power of two we have

$$L(n) \geq C_n \frac{2^n}{n} (1 - o(1)),$$

and if $n - \log n$ is a power of two, almost all functions of $n$ variables are realized more simply than follows from the upper bound for $L(n)$.

In this paper we find a bound for the complexity of the realization of almost all functions of the algebra of logic by the method of cascades. Suppose there is a set $M$ of almost all functions of the algebra of logic and also a function $L_1(n)$ such that if $f(x_1, x_2, \ldots, x_n) \in M$, it is realized by the method of cascades with complexity asymptotically equal to $L_1(n)$. It is shown (Theorems 1, 2) that

$$L_1(n) \sim B_n \frac{2^n}{n},$$

where the approximate form of the graph of the coefficient $B_n$ is shown in Fig. 1. In this paper we consider the two cases separately - when $n - \log n$ is distant from a power of two and when $n - \log n$ is at a "finite distance" from a power of two. We note that it is also possible to prove a bound for $L_1(n)$ in which the two cases are considered together, but the study of the coefficient $B_n$ involves particular difficulties.

The scheme $S$ which realizes $f(x_1, x_2, \ldots, x_n)$ by the method of cascades contains $n + 1$ levels of nodes (the 0th, 1st, $\ldots$, $n$th) and $n$ levels of contacts (the 1st, 2nd, $\ldots$, $n$th). We can show that for almost all functions two contacts leave almost all nodes in an arbitrary realization. We say that the node $P_i$ is degenerate if two contacts do not leave $P_i$. Let $Q_n$ denote the set of all functions of the algebra of logic of $n$ variables for which there is a realiza-
tion $S$ such that at the $l$-th level of $S$ there are more than $(1/\log n)^{2^l}$ degenerate nodes $l \leq n - \log n/2$. We denote the cardinality of the set $Q_n$ by $|Q_n|$.

**Lemma 1.**

$$\log |Q_n| \leq \left(1 - \frac{1}{2\log n}\right) 2^n + O\left(\frac{2^n}{\sqrt{n}}\right).$$

Let us fix the method of realizing functions of $n$ variables by the method of cascades. We expand $f(x_1, x_2, \ldots, x_n)$ in the order $x_1, x_2, \ldots, x_n$. Let us also fix the parameters $l \leq n - \log n/2$, $t \geq \frac{1}{2\log n}$. Let $Q_{a, l, t}$ denote the set of functions of $n$ variables for which at the $l$-th level of the fixed realization there are exactly $t$ degenerate nodes. Let $|Q_{a, l, t}|$ denote the cardinality of $Q_{a, l, t}$. The function $f(x_1, x_2, \ldots, x_n)$ is completely defined by the ordered set

$$(\varphi_0(x_1, x_2, \ldots, x_n), \varphi_1(x_1, x_2, \ldots, x_n), \ldots, \varphi_{2^n-1}(x_1, x_2, \ldots, x_n))$$

of its subfunctions of $n - 1$ variables. To any node of the $l$-th level there corresponds one or more subfunctions of this set. Consequently, there are $t_1 \geq t$ "degenerate" functions for which at least one of the following conditions holds:

1. $\varphi_q(0, x_1, x_2, \ldots, x_n) \equiv 0$;
2. $\varphi_q(1, x_1, x_2, \ldots, x_n) \equiv 0$;
3. $\varphi_q(0, x_1, x_2, \ldots, x_n) \equiv \varphi_q(1, x_1, x_2, \ldots, x_n)$.

Over the functions of the set $(\varphi_0, \varphi_1, \ldots, \varphi_{2^n-1})$ which satisfy condition 1 we place the mark (*), over functions which satisfy condition 2 and are not yet marked we place the mark (**), and over functions which satisfy condition 3 we place the mark (***)

Assume that we know the locations of the marks and the marks themselves. Then the function $f(x_1, x_2, \ldots, x_n)$ is automatically defined on

$$2^t \leq 2^{n-1} \leq 2 \log n$$

of the sets. We find a bound for the number of methods of placing the marks. To do this we have to choose $t_1$ functions of the set $(\varphi_0, \varphi_1, \ldots, \varphi_{2^n-1})$. This can be done in $\binom{2^t}{t_1}$ ways. Marks can be placed over the function in $3^t_1$ ways. Since $l \leq n - \frac{\log n}{2}$ and $2^t \leq 2^{n/\sqrt{n}}$, we find that

$$\log |Q_{a, l, t}| \leq 2^n \left(1 - \frac{1}{2\log n}\right) + O\left(\frac{2^n}{\sqrt{n}}\right).$$

Then

$$|Q_n| \leq n! n \cdot 2^t \cdot |Q_{a, l, t}|,$$

and the lemma is proved.

The number of nodes in the scheme for $f$ corresponds to the number of subfunctions of $f$ for a given decomposition. To estimate the number of subfunctions we use a refinement of Yablonski's method. Let $V_{n,k}$ denote the set of $k$ pairwise distinct variables

$$V_{n,k} = \{x_{a_1}, x_{a_2}, \ldots, x_{a_k}\}, \quad a_j \leq n \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Let $\xi_{V_{n,k}}(f)$ be the number of all subfunctions of $f$ which are obtained by replacing the variables of $V_{n,k}$ by constants. We consider $\xi_{V_{n,k}}(f)$ as a random variable, defined in the space of elementary events of all functions of $n$ variables. Let $M(\xi_{V_{n,k}})$ denote the mathematical expectation of the random variable $\xi_{V_{n,k}}$.

**Lemma 2.**

$$M(\xi_{V_{n,k}}) = 2^{n-k}\left(1 - \left(1 - \frac{1}{2^{n-k}}\right)^{2^k}\right).$$

As in [4] we write down on the left the column of all $2^n$ functions of $n$ variables and on the right the column of all $2^{2^n-k}$ functions of $n - k$ variables. We join the function $f(x_1, x_2, \ldots, x_n)$ of $n$ variables by an arrow with the function $\varphi$ of $n - k$ variables if $\varphi$ is obtained from $f$ by replacing the variables of $V_{n,k}$ by constants. Let $\tau(\varphi)$ be the arrows leading to the function $\varphi$. We find an estimate for $\tau(\varphi)$. From all the functions of $n - k$ variables, apart from $\varphi$ we can form exactly $(2^{2^n-k} - 1)^{2^k}$ distinct functions of $n$ variables. Then