This corollary follows from Theorems 1 and 2 since \( 1 \leq \sup DS_{f(x),\zeta}(x) \leq \sup DS_{f(x)}(x) \leq 1. \)

**COROLLARY 2.** Let \( x = \mu \{ \varepsilon^{-1}(x) \} \), the function \( \omega \) be uniformly differentiable at the point \( x \) along each direction from a set \( U(x) \), \( U'(x) \) and \( T(x) \subseteq U(x) \). Then \( \sup_{i \in T(x)} \omega'(x, i) = \omega'(x, T(x)) = 1. \)

This corollary follows from Corollary 1 and the definition of uniform differentiability along a direction.

**LITERATURE CITED**


**UNCOUNTABLE R-SETS AND N-SETS**

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In this note, we consider problems on whether certain subsets of the real line belong to the classes of R-sets and N-sets. These problems turn out to be unsolvable in the usual ZFS (Zermelo-Frenkel) system of axioms of set theory.

Let us recall the definitions of R-sets and N-sets.

A subset \( E \) of \([0, 1]\) is called an R-set if there exists a trigonometric series

\[
\sum_{n=1}^{\infty} a_n \cos 2\pi n x + b_n \sin 2\pi n x,
\]

that is convergent in \( E \) and whose coefficients do not converge to zero.

A subset \( E \) of \([0, 1]\) is called an N-set if there exists a series of the form (1) that is absolutely convergent in \( E \) and is such that the sum of the moduli of its coefficients is divergent. If a set is not an N-set, then it is called an A.C.-set.

We know [1, pp. 173, 174, 736, 737, 747] that R-sets and N-sets of measure zero and of first category and that each countable set is an R-set as well as an N-set.

Can it be asserted that each set of cardinality less than the continuum \( \mathfrak{c} \) is an R-set and an N-set? This problem turns out to be unsolvable in ZFS. It is interesting that the analogous problem for U-sets (a set \( E \) is called a set of uniqueness or a U-set if each trigonometric series that converges to zero outside \( E \) vanishes identically) has an affirmative solution: Each set of cardinality less than \( \mathfrak{c} \) is a U-set (see [2], [3]).

Under the continuum hypothesis CH, each set of cardinality less than \( \mathfrak{c} \) is a countable set and is, consequently, an R-set as well as an N-set. But the situation is nonunique under negation of CH. On one hand, Hechler [4] has proved the consistency of the statement on the existence of a set of the real line of cardinality less than \( \mathfrak{c} \) that is not of first category with ZFS. Such a subset of \([0, 1]\) is neither an R-set nor an N-set. (Let us observe that the unsolvability of the following problems in ZFS follows from here: Is it necessary that each A.C.-set that is not an R-set has cardinality \( \mathfrak{c} \) (?). On the other hand, we prove that each set of cardinality less than \( \mathfrak{c} \) is an R-set as well as an N-set under the Martin axiom (an additional set-theoretical axiom, consistent with ZFS). In this note we also prove that the union of an N-set and a set of cardinality less than \( \mathfrak{c} \) is an N-set under the Martin axiom.
The Martin axiom (see [5] and [6] is a nontrivial generalization of CH and is consistent
with the negation of CH. We require a series of definitions for the formulation of this axiom.

Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set. Two elements $p$ and $q$ of $\mathcal{P}$ are said to be
compatible if there exists $r \in P$ such that $r \leq p$ and $r \leq q$ in the contrary case, these
elements are said to be incompatible. If each subset of $\mathcal{P}$, that consists of pairwise incompati-
bile elements is countable, then $\mathcal{P}$ is said to satisfy the Suslin conditions. A subset $\mathcal{D}$ of $\mathcal{P}$
is said to be dense in $\mathcal{P}$, if for each $p \in \mathcal{P}$ there exists a $d \in \mathcal{D}$ such that $d \leq p$.

The Martin Axiom MA. Let $\mathcal{P}$ be a partially ordered set that satisfies the Suslin condi-
tion. If $\mathcal{F}$ is a family of dense subsets of $\mathcal{P}$ and the cardinality of $\mathcal{F}$ is less than $\aleph_1$
then there exists an $\mathcal{F}$-generic subset $G$ of $\mathcal{P}$ i.e., a set $G$ that satisfies the following condi-
tions:

1) $F \cap G \neq \emptyset$ for any $F \in \mathcal{F}$;
2) For arbitrary $g$ and $g'$ from $G$ there exists an element $r \in G$ such that $r \leq g$, and
    $r \leq g'$;
3) If $g \in G$ and $g' \geq g$, then $g' \in G$.

Let us observe that the conditions of the Martin axiom are fulfilled under CH, i.e., MA
becomes a theorem under CH.

In the sequel we will need the following definition and statements.

A subset $E$ of $[0, 1]$ is called an $N_0$-set if there exists a sequence of integers $(n_k)$
such that the series $\sum_{k=1}^{\infty} \sin n_k x$ is absolutely convergent in $E$.

We know [1, pp. 737 and 757] that each set of type $N_0$ is also an $R$-set as well as an
$N$-set. The following lemma has been proved in [1, p. 737].

**Lemma 1.** Let $\epsilon_n \to 0$ and suppose that $E$ has the following property: There exists a
sequence of integers $n_k$ such that for each $x \in E$ there exists a $k_x$ (depending on $x$) such
that $|(\epsilon_n x)| < \epsilon_k$ for $k > k_x$ (here $\{t\} = t - v$, where $v$ is the integer nearest to $t$). Then $E$
is an $N_0$-set and is, therefore, an $R$-set as well as an $N$-set.

We also need the following theorem (see [1, p. 903] and also [2]) and a corollary of it
[1, p. 904].

**The Dirichlet-Minkowski Theorem.** Let $t_1, t_2, \ldots, t_v$ be arbitrary numbers. For arbitrary
$A$ there exist an integer $q > A$ and integers $p_1, p_2, \ldots, p_v$, such that
$$|t_i - p_i/q| < 1/q^{1+v} \quad (i = 1, 2, \ldots, v).$$

Hence
$$|\{q t_i\}| < q^{-v} \quad (i = 1, 2, \ldots, v).$$

Corollary of the Dirichlet-Minkowski Theorem: For arbitrary real numbers $t_1, t_2, \ldots, t_v$
and for arbitrary $\epsilon > 0$ there exists an integer $\lambda > 1$, such that
$$|\{\lambda t_i\}| < 2\epsilon \quad (i = 1, 2, \ldots, v) \text{and} \lambda \leq e^{-v}.$$

We will use the following notation. The cardinality of a set $A$ will be denoted by $|A|$.
The letter $N$ will denote the set of natural numbers.

Now we can prove the following theorem.

**Theorem 1 [MA].** Each subset of cardinality less than $\aleph_1$ of the segment $[0, 1]$ is an
$N_0$-set and is therefore an $R$-set as well as an $N$-set.

**Proof.** Let $A$ be a subset of $[0, 1]$ of cardinality less than $\aleph_1$ and $\{\epsilon_n\}$ be a sequence
of positive numbers that converges to zero. Let us consider the set $P$ whose elements are
the ordered pairs of finite subsets of $A$ and $N$, i.e.,
$$P = \{\langle A, K \rangle | A \subseteq A, K \subseteq N, A \text{and } K \text{ are finite}\}.$$ 

Let us order $P$ in the following manner: Let $p_1 = \langle A_1, K_1 \rangle$ and $p_2 = \langle A_2, K_2 \rangle$; set $p_1 \gg p_2$, if and only if 1) $A_1 \subseteq A_2$, and $K_1 \subseteq K_2$; 2) if $K_2 \setminus K_1 \neq \emptyset$, then, writing down the elements of
$K_1$ and $K_2$ in increasing order, we get

*This means that the theorem is proved under the Martin axiom.*