It follows from inequality (18) that

$$2\mu \int_{0}^{b} \|z(t, \varepsilon)\|^2 \, \mathrm{d}t \leq \varepsilon (\|z(a, \varepsilon)\|^2 - \|z(b, \varepsilon)\|^2).$$

Therefore, $u(t, \varepsilon) \to \Phi(\vec{r})$ in $L^2[a, b]$ as $\varepsilon \to 0$. In similar manner we verify that $\tilde{u}(t, \varepsilon) \to \Phi(\vec{r})$ in $L^2[a, b]$ as $\varepsilon \to 0$. Since $\|u(t, \varepsilon) - \Phi(\vec{r})\| \leq \|u(t, \varepsilon) - \tilde{u}(t, \varepsilon)\| + \|\tilde{u}(t, \varepsilon) - \Phi(\vec{r})\|$, it follows that $u(t, \varepsilon) \to \Phi(\vec{r})$ in $L^2[a, b]$ as $\varepsilon \to 0$. By virtue of the uniqueness of limit in $L^2[a, b]$, we have $\Phi(\vec{r}) = \Phi(\vec{r})$. The theorem is proved.

LITERATURE CITED


THE BÄCKLUND TRANSFORMATION AND INTEGRABLE INITIAL BOUNDARY VALUE PROBLEMS

I. T. Khabibullin

UDC 517.958

The method of the inverse scattering problem, a nonlinear analogue of Fourier's method, is an effective way to study nonlinear equations of mathematical physics [1, 2]. The analytical aspect of this method includes the construction of particular solutions of equations and the analysis of the general solution, for example, of the solution of Cauchy problem for a given class of initial values.

A partial differential equation of form

$$u_t = f(u, u_1, \ldots, u_k),$$

where $u = u(x, t)$ is in general a vector-valued function and $u_i = \partial^i u / \partial x^i$, is covered by the method of the inverse scattering problem if it can be written in the form of a zero curvature condition

$$U_t = V_x + [V, U].$$

where $U = U(u, \lambda)$, $V = V(u, u_1, \ldots, u_{k-1}, \lambda)$ are rational matrix-valued functions of a variable $\lambda$, Eq. (2) is a compatibility condition for the following linear system:

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi,$$

where $\Psi = \Psi(x, t, \lambda)$. The inverse problem method is based on the fact that with representation (2) the study of nonlinear equation (1) reduces to the study of the properties of the $\Psi$-function, which is the solution of system of linear equations (3).

Until recently, the study of mixed problems for nonlinear equations on the interval and the semiaxis in the framework of the inverse problem method has been limited to the case of periodic boundary conditions and their variations (see [1, p. 112; 3]). Sklyanin made a significant progress in this area by discovering in 1987 a class of nontrivial boundary conditions that are compatible with the integrability of equations (see [4]).

In this article (see also [5]) we describe a method for constructing integrable initial boundary value problems which is different from the one cited in [4] and which is based on Bäcklund transformations (see [6]), i.e., transformations that map solutions of the equation to other solutions. Let $\tilde{u}(x, t)$ be a solution of Eq. (1) obtained as a result of an application of a Bäcklund transformation to some other solution $u(t, x)$ of the same equation. Then a function $\tilde{u}(x, t)$ defined by

$$\tilde{u}(x, t) = \begin{cases} u(x, t), & x \leq 0, \\ \tilde{u}(x, t), & x > 0, \end{cases}$$

satisfy Eq. (1) for \( x \neq 0 \) and a certain condition on the "gap" at point \( x = 0 \) [see Eq. (1.5) below]. If Eq. (1) and its Bäcklund transformation admit a reduction of reflection type \( \tilde{u}(x, t) = h(u(-x, t)) \) then this condition on the "gap" becomes a boundary value condition compatible with the integrability of the equation. This method of constructing boundary value conditions does not rely on the hamiltonicity of the equation and can be applied, in particular, to Burgers type equations. We construct new examples of boundary value conditions that are compatible with the integrability [see (3.3), (3.5)-(3.7), (3.9), (3.10), (3.12)].

In this paper we also describe an algorithm for reducing an initial boundary value problem on the semiaxis with a boundary condition that is compatible with the integrability of the equation to Cauchy problem on the entire axis.

We note that the problem of boundary values for integrable equations has also been in [7-9].

1. Bäcklund Transformation and a Locally Perturbed Equation. Suppose that Eq. (1) can be written as zero curvature condition (2). By definition (see [10]), two solutions \( u(x, t), \tilde{u}(x, t) \) of Eq. (1) are related by a Bäcklund transformation if eigenfunctions \( \psi \) and \( \bar{\psi} \) of linear system (3) and a linear system
\[
\bar{\psi} = F(\lambda) \psi,
\]
where \( F(\lambda) \) is a matrix polynomial of a parameter \( \lambda \). Eliminating \( \psi, \bar{\psi} \) from Eqs. (3), (1.1), and (1.2), we obtain the following system of equations for the polynomial \( F \):
\[
F_x = \bar{U} F - FU, \quad F_t = \bar{V} F - VF,
\]
The above system can be used to obtain an explicit expression for the coefficients of the polynomial \( F \) in terms of solutions \( u \) and \( \tilde{u} \) of Eq. (1) and to derive the following differential relations between functions \( u(x, t) \) and \( \tilde{u}(t, x) \):
\[
\begin{align*}
\tilde{u}_x &= p(\tilde{u}, u, u_1, \ldots, u_n), \\
\tilde{u}_t &= q(u, u_t, u_1, \ldots, u_n).
\end{align*}
\]
Equation (1.4) is an equivalent definition of the Bäcklund transformation. It is easy to see that the function \( \tilde{u}(x, t) \) defined by Eq. (4) satisfies the following system of relations, which is a local perturbation of Eq. (1):
\[
\begin{align*}
\tilde{u}_t &= f(\tilde{u}, \tilde{u}_t, \ldots, \tilde{u}_t), \\
\tilde{u}_x &= p(\tilde{u}, u, \ldots, u_n),
\end{align*}
\]
where the signs \( \cdot \) denote limits as \( x \) tends to zero from the right and the left. The second condition in (1.5) describes a "gap" in the function \( \tilde{u}(x, t) \) at \( x = 0 \). Locally perturbed Eq. (1.5), like the original Eq. (1), can be written in the zero curvature form. The corresponding linear problem has the following form:
\[
\begin{align*}
\Psi_x &= U \Psi, \quad \Psi_t = V \Psi, \quad x \neq 0, \\
\Psi(0^-, t, \lambda) &= F(0^-, t, \lambda) \Psi(0^+, t, \lambda).
\end{align*}
\]
The factor \( F(0^-, t, \lambda) \) acts as an "admixture" used in [11-12] to integrate initial boundary value problems for nonlinear Schrödinger equations.

2. A Boundary Condition Compatible with the Integrability of the Equation. The locally perturbed equation studied in Sec. 1 is an auxiliary object in the derivation of the boundary condition. Assume that the original equation (1) does not change under a change of variables \( u(x, t) \rightarrow h(u(-x, t)) \) and its Bäcklund transformation is invariant under a pointwise transformation of form \( \tilde{u}(x, t) \rightarrow h(u(-x, t)) \), \( u(x, t) \rightarrow h^{-1}(u(-x, t)) \), where \( h = h(u) \) is a smooth invertible function and \( h^{-1} \) is its inverse. Clearly, in this case the first of relations (1.4) becomes
\[
\tilde{u}_x = p(\tilde{u}, u, u_x),
\]
i.e., the invariance condition implies that in (1.4) we have \( n = 1 \). Therefore, if we require that \( \tilde{u}(x, t) = h(u(-x, t)) \) then the condition on the "gap" in system (1.5) becomes a boundary condition at \( x = 0 \)
\[
u_x = -(h_x(u))^{-1} p(h(u), u, u_x)|_{x=0, t \in \mathbb{R}}.
\]