COMPUTATIONAL ANALYSIS OF SINGLE-SERVER BULK-ARRIVAL QUEUES: $G_l^X/M/1$

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Abstract

This paper deals with numerical computations for the bulk-arrival queueing model $G_l^X/M/1$. First an algorithm is developed to find the roots inside the unit circle of the characteristic equation for this model. These roots are then used to calculate both the moments and the steady-state distribution of the number of customers in the system at a pre-arrival epoch. These results are used to compute the distribution of the same random variable at post-departure and random epochs. Unifying the method used by Easton [7], we have extended its application to the special cases where the interarrival time distribution is deterministic or uniform, and to cases where $X$ has a given arbitrary distribution. We also improved on the various root-finding methods used by several previous authors so that high values of the parameters, in particular large batch sizes, can be investigated as well.

Keywords

Roots, random group arrival, single server, probability distribution.

1. Introduction

For the queueing system $G_l^X/M/1$, the pre-arrival steady-state probability generating function is related to the roots of the characteristic equation of the model (see, for example, Chaudhry and Templeton [5], pp. 158–159). This type of solution, common to many queueing systems, implies that in order to get numerical results of
use to practitioners, one has to calculate those roots. To avoid the difficulties involved in solving the characteristic equation, some authors have concentrated their efforts on finding new methods to evaluate the probability distribution (Neuts [9], Grassmann and Chaudhry [8], Powell [11]). Although easier to use, these methods are iterative: that is, the evaluation of the tail of the distribution creates problems and also high values of the model parameters slows the process down drastically, as reported by Powell [11] and by Chaudhry and Kashyap [4] in their recent study.

In this paper, we develop an algorithm that solves the characteristic equation of systems of the type \( GIX/M/1 \) for the \( k \) roots inside the unit circle, \( k \) being the maximum possible size of an arriving group. The roots are used to invert numerically the probability generating functions so that the state probabilities are evaluated recursively. This procedure has the advantage that it can be stopped at any state without loss of accuracy in the previous probabilities. We extensively tested our algorithms: the program was run for large possible batch sizes (up to 300), for both cases where \( X \) was given a common distributional form such as truncated Poisson or an arbitrary one, and for various values of the traffic intensity (between 0.1 and 0.9). The results obtained enable us to conclude that the algorithms are efficient, fast for at least medium values of \( \rho \), accurate and stable.

In the system \( GIX/M/1 \), the interarrival times of groups are independent identically distributed random variables with distribution function \( A(u) \) and finite mean \( 1/\lambda \). A single server serves customers one at a time; the service times are independent exponentially distributed with finite mean \( 1/\mu \). The group size \( X \) is a random variable with \( \Pr(X = i) = g_i \) (\( i = 1, 2, \ldots, k \)) and mean \( \overline{g} = \sum g_i \). We assume that the utilization factor \( \rho = \overline{g} \lambda/\mu < 1 \). For this model, it is shown in Chaudhry and Templeton [5] that

\[
P^{-}(z) = \sum_{n=0}^{\infty} P^{-}\!_n z^n = \prod_{i=1}^{k} \frac{1 - \gamma_i}{1 - z \gamma_i} \quad (|z| \leq 1), \tag{1.1}
\]

where \( P^{-}\!_n \) is the pre-arrival steady-state probability that there are \( n \) customers in the system, and \( \gamma_1, \gamma_2, \ldots, \gamma_k \) are the \( k \) roots inside the unit circle of the characteristic equation (c.eq.) for the model. The c.eq. has the form

\[
G(z^{-1})K(z) = 1, \tag{1.2}
\]

with \( G(z) = \sum g_i z^i \) and

\[
K(z) = \int_0^\infty e^{-\mu(1-z)u} A(u) \, du = \overline{a}(\mu - \mu z),
\]