INFLUENCE OF DEFORMATION OF THE BOUNDARY ON CERTAIN CONFORMAL PROPERTIES OR REGIONS (REGIONS IN V. I. SMIRNOV'S CLASS $S$)

G. Ts. Tumarkin

The influence of deformation of the boundary of a region on its belonging to V. I. Smirnov's class $S$ is investigated.

V. I. Smirnov [1, 2] and M. V. Keldysh and M. A. Lavrent'ev [3] have proved that many problems concerning boundary properties of analytic functions in regions $G$ with a rectifiable Jordan boundary $\gamma$, and also problems concerning the approximation by polynomials on $\gamma$, are related to whether $G$ satisfies V. I. Smirnov's condition. This condition is as follows: let $z = \varphi(w)$ map $|w| < 1$ conformally on $G$. Then we say that $G$ satisfies V. I. Smirnov's condition if $\ln |\varphi'(w)|$ can be represented in the disk $|w| < 1$ by the Poisson-Lebesgue integral:

$$\ln |\varphi'(re^{i\theta})| = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} \ln |\varphi'(e^{i\phi})| d\phi.$$

(1)

The class of such regions will be denoted by $S$. We note that the condition that $G$ be a Jordan region with a rectifiable boundary implies that $\varphi'(w)$ is in the class $\Pi_1$, and so $\ln |\varphi'(w)|$ can be expressed by the Poisson-Stieltjes integral. M. V. Keldysh and M. A. Lavrent'ev [3] have constructed a remarkable example of a region with a rectifiable Jordan boundary which does not belong to $S$ (see [5], p. 229). New examples of this type are also given in [4]. Since the formulation of many results are much simpler for regions $G$ of $S$ than for regions $G \notin S$ (see [5], p. 238), it is desirable to have a rather general criterion for a region's belonging to $S$. Criteria directly related to the geometrical properties of the boundary $\gamma$ of $G$ would be particularly interesting. Conditions of this type are discussed in works by V. I. Smirnov, M. V. Keldysh, M. A. Lavrent'ev, the author [6], and G. Shapiro [7].

In the present work we study the relation between deformations of the boundary $\gamma$ of the region $G$ and the property that the region belongs to V. I. Smirnov's class $S$. Here we consider only "internal" deformations, in which a countable number of arcs $\gamma_k$ of the boundary $\gamma$ of $G$ are replaced by arcs $\tilde{\gamma}_k$ with the same ends as $\gamma_k$, but located entirely inside $G$. We prove that, if $G$ is in $S$, then any region $\tilde{G} \subseteq G$, obtained by "internal" deformations of $\gamma$ by "good" (Smirnov) arcs is also in $S$. On the other hand if $G$ is not in $S$, there is a set $E_0$ of measure zero on $\gamma$ such that, if we take any open set $O \supseteq E_0$ and carry out the internal deformation process described above for the arcs $\gamma_k$ forming $O$ to obtain arcs $\tilde{\gamma}_k$, then the resulting region $\tilde{G}$ will belong to $S$. It is of course of interest to consider not only "internal" but also "external" deformations of $\gamma$. In the present state of knowledge, however, this would be a very laborious task.

---

*We use the term Smirnov arc for an arc which can be closed by an auxiliary arc to form a region in $S$ (for more details see [6]). It follows from known results that arcs $\Gamma$ with any one of the following properties possess the above property: 1) $\Gamma$ is smooth; 2) $\Gamma$ is of finite rotation; 3) the ratio of the length of any arc $\gamma \subset \Gamma$ to the length of the chord is bounded. We will use the same symbol $S$ to denote the class of Smirnov arcs.

THEOREM 1. Let $G \subset S$. Then every subregion $\tilde{G} \subset G$ with a rectifiable Jordan boundary, possessing the property that every part of its boundary $\tilde{\gamma}$ in the interior of $G$ consists of Smirnov arcs $\tilde{\gamma}_k$:

$$\tilde{\gamma} \cap G = \bigcup_{k} \tilde{\gamma}_k \subset S,$$

is also in the class $S$.

Proof. Let $\tilde{D}$ be the subregion of the disk $|\tilde{w}| < 1$, corresponding to $\tilde{G}$ under the mapping $z = \varphi(\tilde{w})$, and let $\tilde{\omega} = \omega(w)$ map $|w| < 1$ conformally onto $\tilde{D}$. Then $z = \varphi(\omega(w))$ maps $|w| < 1$ conformally onto $\tilde{G}$.

Letting $z = \varphi(\omega(w)) = \varphi(w)$:

$$\ln |\varphi'(w)| = \ln |\varphi'| \omega(w)| + \ln |\omega'(w)|$$

(2)

We will have proved that $\tilde{G}$ belongs to $S$ if we establish that each of the terms on the right in (2) can be represented by a Poisson-Lebesgue integral. For the function $\ln |\varphi'| \omega(w)|$, this is a direct consequence of the following considerations. The function $\ln |\varphi'|(w)|$ can be expressed in $|w| < 1$ by a Poisson-Lebesgue integral, since $G \subset S$. But it is known that if a harmonic function can be expressed by a Green's integral in a region, then it possesses the same property in a subregion (see [2]). Thus $\ln |\varphi'|(w)|$ can be represented in $\tilde{D}$ by a Green's integral. Using the fact that representability by Green's formula is conformally invariant, we conclude that $\ln |\varphi'| \omega(w)|$ can be represented by the Poisson-Lebesgue formula in $|w| < 1$.

It remains to verify that $\ln |\omega'(w)|$ can be represented by a Poisson-Lebesgue integral in $|w| < 1$.

Assume the contrary. The function $\ln |\omega'(w)|$ is, in view of (2), equal to the difference of two harmonic functions. The function $\ln |\varphi'(w)|$ can plainly be expressed by a Poisson-Stieltjes integral

$$\ln |\varphi'(w)| = \int \frac{1}{2\pi} \sum_{\theta} \frac{1}{1 + r^2 - 2r \cos (\theta - \omega)} d\mu(\theta)$$

with a function $\mu(\theta)$ whose singular component $\mu_S(\theta)$ is an increasing function:

$$d\mu(\theta) = d\mu_\alpha(\theta) + d\mu_\beta(\theta) = \mu^* (\theta) d\theta + d\mu_\beta(\theta)$$

$$d\mu_\beta(\theta) \leq 0$$

(see [5], p. 220). Since we have proved that $\ln |\varphi| \omega(w)|$ can be represented by a Poisson-Lebesgue integral, the function $\ln |\omega'(w)|$ can be represented by a Poisson Stieltjes integral:

$$\ln |\omega'(w)| = \int \frac{1}{2\pi} \sum_{\theta} \frac{1}{1 + r^2 - 2r \cos (\theta - \omega)} d\nu(\theta)$$

(3)

and $d\nu_S(\theta) = d\mu_S(\theta) \leq 0$. Suppose that $d\mu_S \neq 0$. It is proved in [6] that there are no points of increase of the singular component $d\nu_S(\theta)$ on arcs $\tilde{\gamma}_k$ of the circle $|w| = 1$, corresponding by the conformal mapping $z = \varphi(w)$ to Smirnov arcs $\tilde{\gamma}_k$. Hence the support of the measure $d\nu_S(\theta)$ must lie on the set of points of $|w| = 1$, corresponding under the conformal mapping to points of $\tilde{\gamma}$, and so to points $\tilde{w}$ with $|\tilde{w}| = 1$ for the mapping $\tilde{w} = \omega(w)$. If $d\nu_S(\theta) = 0$, then De la Vallée-Poussin's theorem ([8], p. 195) implies that there is a point $\Theta(\theta)$ such that $\nu^* (\Theta) = -\infty$. Then $\nu^* (\Theta)$ tends to $\Theta$ we obtain

$$\lim |\omega'(w)| = -\infty,$$

whence $|\omega'(r)e^{i\theta}| = 0$ for $r \to 1$. It follows from this that the distance between the points $\tilde{w}_r$ and $\tilde{w}_0$, corresponding to $e^{i\theta}$ under the mapping $\tilde{w} = \omega(w)$, must be of order $o(1-r)$ for $r \to 1$:

$$|\tilde{w}_r - \tilde{w}_0| \leq \frac{1}{2\pi} \int |\omega'(re^{i\theta})| d\theta = o(1-r)$$

(3)

But Schwartz's lemma implies that, since $|\omega(w)| < 1$ in $|w| < 1$, we have $|\omega(w)| \leq r$ for $|w| \leq r$, and so the distance between $\tilde{w}_r$ and $\tilde{w}_0$, measured on $|\tilde{w}| = 1$, satisfies the inequality

$$|\tilde{w}_r - \tilde{w}_0| \geq 1 - r.$$  

(4)

The contradiction between (4) and (3) proves that the assumption $d\mu_S(\theta) \neq 0$ was false.

Theorem 2 gives a result concerning the possibility of obtaining a region $\tilde{G}$ in the class $S$, by "internal" deformations of arcs $\gamma_k$ of the boundary $\gamma$ of a region $G \not\subset S$, such that the sum of the lengths of the $\gamma_k$ is smaller than $\gamma_k < \varepsilon$.

THEOREM 2. Let $G \not\subset S$. Then there is a set $\varepsilon$ of zero measure on the boundary $\gamma$ of $G$ such that, if we take any open set $O \supset \varepsilon$ and replace each of the arcs $\gamma_k$ forming $O$ by a rectifiable arc $\tilde{\gamma}_k \subset G$, with the