THE MODULUS OF CONTINUITY IN $L_p$

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Introduction. Let $1 \leq p \leq \infty$, $L_p$ be the space of the $p$-th power-integrable 1-periodic functions $f(x)$ with the norm

$$||f||_p = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \quad (1 \leq p < \infty),$$

$L_\infty = C$ be the space of the continuous 1-periodic functions with the norm

$$||f||_C = \sup |f(x)|;$$

$$\omega(\delta, f)_p = \sup \{|f(x+t) - f(x)|: 0 \leq t \leq \delta\} \quad (0 < \delta < 1/2)$$

be the modulus of continuity of $f(x)$ in $L_p$.

We know that if $\omega(\delta)$ is a modulus of continuity in $L_p$, then

$$\omega(\delta) \rightarrow 0 \quad (\delta \rightarrow 0),$$

$$0 \leq \omega(\delta_2) - \omega(\delta_1) \leq \omega(\delta_2 - \delta_1) \leq M \quad (0 < \delta_1 < \delta_2),$$

Lebesgue [1] has observed that each function $\omega(\delta)$ with the indicated properties is the modulus of continuity of a certain continuous periodic function. Nikol'skii [2] has observed that we can take $\omega(x)$ as such a function. By the same token, we have obtained a description of the set of moduli of continuity in $C$. Besov and Stechkin [3] have described the set of moduli of continuity in $L_2$ and have shown that it is a proper subset of the set of moduli of continuity in $C$. For the spaces $L_p$, $p \neq 2$, the description of the moduli of continuity has not been obtained. Radoslavov [4] has proved that the moduli of continuity in $C$ are the orders of the moduli of continuity in $L_p$ ($1 \leq p < \infty$). It is interesting to find simple necessary and simple sufficient conditions for moduli of continuity in $L_p$.

Yudin [5] has proved that

$$\omega^2 \left( \frac{1}{2}, f \right)_2 \leq \frac{4}{\pi^2} \int_0^{1/2} \omega^2(t, f)_p \, dt.$$
for \( f \in L_2 \). Konyagin [6] has shown that for each \( f \in L_p \) (\( 1 \leq p \leq 2 \)) there exists a \( g \in L_2 \) such that

\[
\omega^p(\delta, f)_p = \omega^2(\delta, g)_2.
\]

Therefore, for \( f \in L_p \) (\( 1 \leq p \leq 2 \))

\[
\omega^p\left(\frac{1}{2}, f\right)_p \leq \frac{8}{\pi^2} \int_0^{1/2} \omega^p(t, f)_p dt.
\]

Let us observe that (1) can also be obtained from the inequality

\[
\left\| f(x) - \int_0^1 f(x) dx \right\|_p \leq 2^{1/2} \left( \int_0^1 |f(x) - f(y)|^p dx dy \right)^{1/p}
\]

\((1 \leq p \leq 2, 1/p + 1/p' = 1).\)

established by N. I. Chernykh (a communication to the Saratov Winter School on Theory of Functions and Approximation in January 1984) with the help of the numerical inequality

\[
|a - b|^p \geq 2^{p-1} (|a| + |b|)^p - a |b|^{p-1} \text{sgn} b - b |a|^{p-1} \text{sgn} a \quad (1 \leq p \leq 2).
\]

The author proves the following theorem.

**THEOREM 1.** For functions \( f \in L_p \) (\( 2 < p < \infty \))

\[
\omega^p\left(\frac{1}{2}, f\right)_p \leq \frac{8}{\pi^2} \int_0^{1/2} \omega^p(t, f)_p dt \quad \left( \frac{1}{p} + \frac{1}{p'} = 1 \right).
\]

Inequalities (1) and (2) give simple necessary conditions for moduli of continuity in \( L_p \). For example, it follows from (1) and (2) that the function \( \omega(\delta) = \delta^2 \) is not a modulus of continuity in \( L_p \) for \( 1/p < \alpha \leq 1 \) (\( 1 < p \leq 2 \)) and \( 1 - 1/p < \alpha \leq 1 \) (\( 2 < p < \infty \)).

It follows from the results of [3] (see also [7]) that if \( \omega(\delta) \) is a modulus of continuity in \( C \) and \( \omega^2(\delta) \) is an upward-convex function, then \( \omega(\delta) \) is a modulus of continuity in \( L_2 \). In particular, this condition is fulfilled for the function \( \omega(\delta) = \delta^2 \) for \( 0 < \alpha \leq 1/2 \). With the help of the function

\[
g(\omega, t) = \begin{cases} 1/4 \cdot \omega'(1/2 - 2t), & 0 \leq t < 1/4, \\ 1/4 \cdot \omega'(2t - 1/2), & 1/4 \leq t \leq 1/2, \\ \end{cases}
\]

we can easily verify that each upward-convex modulus of continuity \( \omega(\delta) \) in \( C \) is a modulus of continuity in \( L \) also.

The character of the inequalities (1) and (2) as sufficient conditions for moduli of continuity in \( L \) and \( L_2 \) enables us to formulate the following problem:

**Problem.** To show that a modulus of continuity \( \omega(\delta) \) in \( C \) is a modulus of continuity in \( L_p \) if \( \omega^p(\delta) \) is an upward-convex function for \( 1 < p < 2 \) and \( \omega^p(\delta) \) is an upward-convex function for \( 2 < p < \infty \).

1. **Proof of Theorem 1.** The idea of the proof of inequality (2) goes back to [8].

Let \( L_{pr} \) (\( 1 \leq p, r \leq \infty \)) be the space of the 1-periodic (with respect to each argument) functions \( g(x, y) \) with the norm

\[
\| g \|_{pr} = \left( \int_0^1 \left( \int_0^1 |g(x, y)|^p dx \right)^{r/p} dy \right)^{1/r}.
\]

Let us consider the operator

\[
T : L_q \rightarrow L_{pr} \quad (p < q).
\]