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On Equivariant Isometric Embeddings

John Douglas Moore[1,*] and Roger Schlafly[2]

[1] Department of Mathematics, University of California, Santa Barbara, California 93106, USA
[2] Department of Mathematics, University of California, Berkeley, California 94720, USA

1. Introduction

The Nash embedding theorem [13] asserts that any Riemannian manifold possesses an isometric embedding into a Euclidean space of sufficiently large dimension. This article is devoted to a proof of an equivariant version of Nash's theorem.

Main Theorem. *If M is a compact Riemannian manifold and G is a compact Lie group which acts on M by isometries, there is an orthogonal representation ρ of G on some Euclidean space $\mathbb{E}^N$ and an isometric embedding from M into $\mathbb{E}^N$ which is equivariant with respect to ρ.*

The representation ρ can be regarded as a Lie group homomorphism from G into the orthogonal group $O(N)$ which acts on $\mathbb{E}^N$ by rotations and reflections; a smooth map $X\colon M \to \mathbb{E}^N$ is equivariant with respect to ρ if and only if $X(\sigma p) = \rho(\sigma)\, X(p)$, for all $\sigma \in G$, $p \in M$.

The main theorem is true in both the C^∞ and real analytic categories. We will work in the C^∞ category for the time being, and return to the real analytic case in §4. Moreover, the theorem holds for manifolds with boundary.

The main analytic tool used by Nash to prove his isometric embedding theorem is an implicit function theorem based upon the Newton iteration method. The implicit function theorem applies to the equivariant case with virtually no change. In order to apply the implicit function theorem we need to approximate a given G-invariant Riemannian metric on M by a metric induced by an equivariant embedding; we will do this by using the theory of the Laplace operator on compact Riemannian manifolds.

According to Gromov and Rokhlin [7], any n-dimensional compact Riemannian manifold can be isometrically embedded in $\mathbb{E}^N$, where $N = (1/2)\, n(n+1) + 3n + 5$. No such universal bound is possible in the equivariant case, and in fact, given any positive integer N, it is possible to construct a left invariant

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metric on the group S^3 of unit quaternions which is not induced by any equivariant embedding in $\mathbb{E}^n$ for $n \leqq N$, as described in §6 of this article.

Moreover, the equivariant isometric embedding theorem does not hold without the assumption that M be compact. Indeed, Bieberbach [1] shows that the Poincaré disc with the hyperbolic metric of constant curvature -1 together with the circle group of rotations about the origin possesses no equivariant isometric embedding in any finite-dimensional Euclidean space.

It suffices to prove the equivariant isometric embedding theorem in the special case where M is an n-dimensional sphere S^n with a Riemannian metric invariant under a Lie subgroup G of $O(n+1)$. Indeed, by a theorem of Mostow and Palais [2, p. 315], any compact G-manifold possesses an equivariant embedding in a sphere S^n of sufficiently large dimension, even if the manifold has boundary, and by a partition of unity argument one easily extends a G-invariant metric on M to a G-invariant metric on S^n which makes this equivariant embedding isometric.

It will be convenient to formulate the equivariant isometric embedding problem in terms of certain Fréchet spaces. If G is a given compact Lie group acting on a compact manifold M and $\rho: G \to O(N)$ is a given representation, let

$C^\infty(M, \mathbb{E}^N) = \{C^\infty \text{ maps } X: M \to \mathbb{E}^N\}$,

$C^\infty_{G,\rho}(M, \mathbb{E}^N) = \{X \in C^\infty(M, \mathbb{E}^N) \mid X \text{ is equivariant with respect to } \rho\}$,

$\mathrm{Met}^\infty(M) = \{C^\infty \text{ symmetric rank two covariant tensors on } M\}$,

$\mathrm{Met}^\infty_G(M) = \{g \in \mathrm{Met}^\infty(M) \mid g \text{ is } G\text{-invariant}\}$.

These vector spaces become Fréchet spaces with the usual family of C^k norms. (Notation: $\|X\|_k$ and $\|g\|_k$ will denote the C^k norms of elements $X \in C^\infty(M, \mathbb{E}^N)$, $g \in \mathrm{Met}^\infty(M)$.) Finally let

$C^\infty_G(M, \mathbb{E}^N) = \cup \{C^\infty_{G,\rho}(M, \mathbb{E}^N) \mid \rho: G \to O(N) \text{ a representation}\}$.

We define a map $F: C^\infty_G(M, \mathbb{E}^N) \to \mathrm{Met}^\infty_G(M)$ by letting $F(X)$ be the metric induced on M by X. In terms of local coordinates $(u^1, \dots, u^n)$ defined on an open subset U of M,

$$F(X) \mid U = \sum_{i,j=1}^{n} \frac{\partial}{\partial u^i}(X) \cdot \frac{\partial}{\partial u^j}(X)\, du^i\, du^j.$$

To prove the theorem, we need to show that given a positive-definite $g \in \mathrm{Met}^\infty_G(M)$, there is some representation $\rho: G \to O(N)$ and some embedding $X \in C^\infty_{G,\rho}(M, \mathbb{E}^N)$ such that $F(X) = g$.

We say that an element $g \in \mathrm{Met}^\infty_G(M)$ is *realizable* if there is a mapping (not necessarily an embedding) $X \in C^\infty_G(M, \mathbb{E}^N)$ for some N such that $F(X) = g$. The set of realizable metrics is closed under addition and multiplication by positive scalars.

There are two steps to the proof of the equivariant isometric embedding theorem. The first step consists of constructing a specific "perturbable" embedding $X_0 \in C^\infty_G(M, \mathbb{E}^{N_1})$ such that if $g_0 = F(X_0)$, then any $g_1 \in \mathrm{Met}^\infty_G(M)$ which is