1. One of the most fruitful applications of the Phragmén-Lindelöf theorem occurs in analytic number theory in the discussion of the Riemann zetafunction. In its most frequently used form the theorem is applied to a half-strip and to a single function (cf. Landau, Vorlesungen über Zahlentheorie, vol. 2, Satz 404, 405, pp. 48—51). This, however, is not quite general enough for situations encountered e.g. in the theory of character sums and of primes in arithmetic progressions. We need here estimations which are uniform in several parameters for an infinite set of \( L(s, \chi) \) or \( \zeta(s, \lambda) \)-functions, and the whole strip must be considered\(^1\). It is the purpose of the present paper to provide a suitable refinement of the Phragmén-Lindelöf theorem. The improvement is achieved through the use of certain subharmonic functions. The applications in Theorems 3, 4, 5 concern zetafunctions. But already the results about the \( \Gamma \)-function in Lemmas 1 to 3 are believed to be new.

2. Basic for our arguments is the following

**Theorem 1.** Let \( a, b, Q, \gamma, \delta \) be real numbers,

\[
- Q < a < b, \quad \gamma \leq \delta.
\]

Then there exists an analytic function \( \varphi(s) = \varphi(s; Q) \), depending also on the parameters \( a, b, \gamma, \delta \), which is regular in the strip

\[
S(a,b): \quad a \leq \Re(s) \leq b
\]

and such that

\[
\begin{align*}
|\varphi(a + it; Q)| &= |Q + a + it|^{\gamma} \\
|\varphi(b + it; Q)| &= |Q + b + it|^{\delta}
\end{align*}
\]

and that for \( a \leq \sigma \leq b \)

\[
|\varphi(s; Q)| \geq |Q + s|^{l(\sigma)}
\]

where

\[
l(\sigma) = \gamma \frac{b - \sigma}{b - a} + \delta \frac{\sigma - a}{b - a}.
\]

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\(^1\) Such a situation is found e.g. in the discussion of the Hecke functions in [5], Hilfssatz 15, pp. 363—365, where, however, a rather crude treatment if sufficient.
Moreover
\begin{equation}
\varphi(s; Q) = O(|t|^c), \quad |t| \to \infty
\end{equation}
for a certain $c > 0$.

\textit{Remark.} Because of (2.1) we have $|Q + s| > 0$ so that (2.3) implies
\begin{equation}
\varphi(s; Q) \neq 0.
\end{equation}
It is essential for our further arguments that we have equality in (2.2) and that (2.3) holds for all $Q$ with $Q + a > 0$.

\textit{Proof.} The case $\gamma = \delta$ is trivial since here $\varphi(s; Q) = (Q + s)^\gamma = (Q + s)^\delta$ furnishes the solution. Here, and in similar situations later, we take the principal branch of $(Q + s)^\gamma$ in $S(a, b)$, i.e. that one which is real for positive $Q + s$. We can now assume $\gamma < \delta$.

We construct a harmonic function $u(\sigma, t)$ in the strip $S(a, b)$ which fulfills the boundary conditions
\begin{equation}
\begin{cases}
u(a, t) = \gamma \log |Q + a + it| = \frac{1}{2} \gamma \log ((Q + a)^2 + t^2) \\
u(b, t) = \delta \log |Q + b + it| = \frac{1}{2} \delta \log ((Q + b)^2 + t^2).
\end{cases}
\end{equation}
The boundary problem for $Au = 0$ in a strip is solved by a transformation of the Poisson formula for the circle. Let us put
\begin{equation}
\Delta (\sigma, t) = \frac{1}{2} \sin \pi \sigma \cosh \pi t - \cos \pi \sigma.
\end{equation}
Then
\begin{equation}
u(\sigma, t) = \frac{1}{b - a} \int_{-\infty}^{\infty} \Delta (\sigma - a, \frac{t - y}{b - a}) A(y) \, dy + \frac{1}{b - a} \int_{-\infty}^{\infty} \Delta (\sigma - a, \frac{t - y}{b - a}) B(y) \, dy
\end{equation}
satisfies $Au = 0$ in the interior of $S(a, b)$ with the boundary conditions
\begin{equation}
u(a, t) = A(t), \quad \nu(b, t) = B(t),
\end{equation}
where $A(t), B(t)$ are supposed to be continuous and $e^{-\pi |a| |b-a|} |A(t)|, e^{-\pi |a| |b-a|} |B(t)|$ integrable in the interval $(-\infty, \infty)$. (Cf \cite{2}, p. 550.)

Taking
\begin{equation}
A(t) = \gamma \log |Q + a + it|, \quad B(t) = \delta \log |Q + b + it|,
\end{equation}
we have in (2.52) a solution of the boundary problem (2.4).

From (2.51), (2.52) (2.6) it follows by a simple estimation that
\begin{equation}
u(\sigma, t) = O(\log (2 + |t|))
\end{equation}
in the strip $S(a, b)$ uniformly in $\sigma$, although not in $Q$. 